Reminder of Random Variables
Indicator RV

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7. Indicator RVs

The **indicator random variable I{A}** associated with **event A** is defined as

$$\text{I}\{A\}=\begin{cases} 1 & if\ \ A\ occurs \\ 0 & if\ A\ does\ not\ occur \end{cases} \qquad (7.1)$$

Example: determine the expected number of heads in tossing a fair coin. **sample space** is

S={H,T}, with Pr{T}= Pr{H}= $\frac{1}{2}$.

Define the event *H* as the coin coming up heads, we define an indicator RV X_H associated with the **event *H***, such that:

X_H counts the number of heads obtained in this flip, i.e. it is 1 if the coin comes up heads and 0 , otherwise.

We write

$$X_H=\text{I}\{H\}=\begin{cases} 1 & if\ \ H\ occurs \\ 0 & if\ T\ occurs \end{cases}.$$

7. Indicator RVs

The expected number of heads obtained in one flip of the coin is simply the expected value of indicator variable X_H:

$$E[X_H] = E[I\{H\}]$$

$$= 1. \Pr\{H\} + 0. \Pr\{T\}$$

$$= 1. \left(\frac{1}{2}\right) + 0. \left(\frac{1}{2}\right) = \frac{1}{2}$$

Thus the expected number of heads obtained by one flip of a fair coin is 1/2.

Q: what is the difference between expected value and average case? Does make sense to define average with one flip ?

7. Indicator RVs

Lemma 7.1

Given a sample space S and an event A in the sample space S, let
$X_A = I\{A\}$.

Then $$E[X_A] = Pr\{A\}$$

Proof:

By the definition of an indicator RV from equation (7.1) and the definition of expected value, we have

$$\begin{aligned} E[X_A] &= E[I\{A\}] \\ &= 1.\Pr\{A\} + 0.\Pr\{\bar{A}\} \\ &= \Pr\{A\} \end{aligned}$$

,where $\bar{A}$ denotes S - A, (i.e. the complement of A).

Thus the above lemma implies:

The expected value of an indicator RV associated with an event A is equal to the probability that A occurs.

7. Indicator RVs

Although indicator RVs may seem cumbersome for an application such as counting the expected number of heads on a flip of a single coin, they are useful for analyzing situations in which we perform repeated random trials.

Example: compute the expected number of heads in *n* tossing of a coin.

Let X denotes the total number of heads in the *n* coin flips, so that

$$X = \sum_{i=1}^{n} X_i$$

we take the expectation of both sides

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} \frac{1}{2}$$

$$= \frac{n}{2}$$

7. Indicator RVs

We can compute the expectation of a random variable having a binomial distribution from equations

$$C_n^k = \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

And

$$f_{n,p}(k) = C_n^k p^k (1-p)^{n-k}$$

$$\sum_{k=0}^{n} f_{n,p}(k) = 1.$$

7. Indicator RVs

Let X~Bin(n; p), q=1-p,   By the definition of expectation, we have

$$E[X] = \sum_{k=0}^{n} k.\Pr\{X = k\}$$

$$= \sum_{k=0}^{n} k.f_{n,p}(k)$$

$$= \sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k} = \sum_{k=1}^{n} k\frac{n}{k}\binom{n-1}{k-1}p^k q^{n-k}$$

$$= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}q^{n-k}$$

$$= np\sum_{k=0}^{n-1}\binom{n-1}{k}p^k q^{(n-1)-k}$$

$$= np\sum_{k=0}^{n-1} f_{n-1,p}(k)$$

$$= np$$

$$k - 1 = j \qquad = k$$

$$k = j + 1$$

$$n - k = n - (j + 1)$$

$$= n - 1 - j$$

7. Indicator RVs

Let X~Bin(n; p), q=1-p. Obtaining the same result using the linearity of expectation.

Let X_i denotes the number of successes in the *i* th trial. Then

$$E[X_i] = p.1 + q.0 = p$$

and by linearity of expectation, the expected number of successes for *n* trials is

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$
$$= \sum_{i=1}^{n} E[X_i]$$
$$= \sum_{i=1}^{n} p$$
$$= np$$

7. Indicator RVs

Example: Let X~Bin(n; p), q=1-p calculate the variance of the distribution. Using $Var[X] = E[X^2] - E^2[X]$.,

we have $Var[X_i] = E\left[X_i^2\right] - E^2[X_i]$.

X_i only takes on the values 0 and 1, we have $X_i^2 = X_i$,

which implies $E\left[X_i^2\right] = E[X_i] = p$.

Hence, $Var[X_i] = p - p^2 = pq$

To compute the variance of X, we take advantage of the independence of the *n* trials; thus,

$$Var[X] = Var\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} Var[X_i]$$

$$= \sum_{i=1}^{n} pq$$

$$= npq$$

تفاوت نشانگر و برنولی ← معمولا متغیر تصادفی با دو حالت است و فقط احتمال رخ دادن دارد
← نشانگر ترکیب فضای نمونه و احتمال هاست / یعنی همه زیرمجموعه‌های فضای نمونه را در نظر می‌گیرد

(برای)

متغیر تصادفی نشانگر (Indicator RV): کمیت‌ها به A یک فضای (نمونه) را به زیرمجموعه‌ها

با نقطه اجتماع و شامل (collectively exhaustive ≠ همه فضا را در بر بگیرد) افراز گفته.

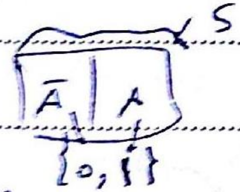
نشانگر پیشامد A را با متغیر تصادفی I_A نشان می‌دهیم و به صورت زیر تعریف می‌کنیم

$$I_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \in \bar{A} \end{cases}$$

کاربرد متغیر تصادفی نشانگر: در مسائلی که روش مستقیم کار کردن وجود ندارد، با مفهوم نشانگر با

استفاده از آن، محاسبات شمارش‌ها را انجام می‌دهیم

پیشامد A فقط وقتی رخ می‌دهد که $I_A = 1$. در مثال $\frac{۲-۱۶}{۸۲}$ تمرین را با این مثال ببینید.

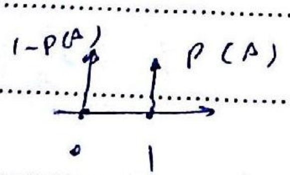


pmf (تابع جرم احتمال) متغیر تصادفی I_A (چون متغیر تصادفی ما دو مقدار ۰ و ۱ می‌گیرد) برنولی است

به صورت زیر بیان می‌شود

$$P_{I_A}(0) = P(\bar{A}) = 1 - P(A)$$

$$P_{I_A}(1) = P(A)$$



https://www.statlect.com/fundamentals-of-probability/indicator-functions

8. Linearity of Expectation

Linearity of Expectation

- Thm. $$E(X_1 + X_2) = \sum_{s \in S} p(s)(X_1(s) + X_2(s))$$

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$$

Very useful result. Holds even when X_i's are dependent!

Proof: (case n=2)

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$$E(X_1 + X_2) = \sum_{s \in S} p(s)(X_1(s) + X_2(s))$$

$$= \sum_{s \in S} p(s) X_1(s) + \sum_{s \in S} p(s) X_2(s)$$

$$= E(X_1) + E(X_2)$$

QED

(defn. of expectation; summing over elementary events in sample space)

Aside, alternative defn. of expectation:

$$E(X) = \sum_{x} x P(X = x)$$

Example

- Consider ***n*** children of different heights placed in a line at random.
- Starting from the beginning of the line, select the first child. Continue walking, until you find a taller child or reach the end of the line.
- When you encounter taller child, also select him/her, and continue to look for next tallest child or reach end of line.
- **Question: What is the expected value of the number of children selected from the line??**

Hmm. Looks tricky…

What would you guess? Lineup of 100 kids... [e.g. 15 more or less?]
Lineup of 1,000 kids… [e.g. 25 more or less?]

- Let *X* be the r.v. denoting the number of children selected from the line.

$$X = X_1 + X_2 + \ldots + X_n$$

where

$$X_i = \begin{cases} 1 & \text{if the tallest among the first i children.} \\ & \text{(i.e. will be selected from the line)} \\ 0 & \text{otherwise} \end{cases}$$

By linearity of expectation, we have:

$$E(X) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

- **What is E(X_i)?**

What is P(X_1 = 1)? A: 1

What is P(X_n = 1)? A: 1/n

What is P(X_i = 1)? A: 1/i

Note that the one tallest person among ***i*** persons needs to be at the end. (by symmetry: equally likely in any position)

Now, E(X_i) = 0 * P(X_i = 0) + 1 * P(X_i = 1)

= 0 * (1 – 1/i) + 1 * (1/i)

= 1/i.

So, E(X) = 1 + 1/2 + 1/3 + 1/4 … + 1/n

$$\approx \ln(n) + \gamma + \frac{1}{n}, \quad where\ \gamma = 0.57722...\ Euler's\ const$$

Consider doubling queue:
What's probablility tallest kid in first half

A: ½
(in which case 2^{nd} half doesn't add anything!)

Intriguing!

Surprisingly small!!

Intuition??

e.g. N = 100 E(X) ~ 5

N = 200 E(X) ~ 6

N = 1000 E(X) ~ 7

N = 1,000,000 E(X) ~ 14

N = 1,000,000,000 E(X) ~ 21

Indicator Random Variable

- Recall: Linearity of Expectation

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$$

Y→ An indicator random variable is:

- 0/1 binary random variable.
- 1 corresponds to an event E, and 0 corresponds to the event did not occur
- Then **E(Y) = 1*P(E) + 0*(1-P(E)) = P(E)**

Expected number of times event occurs:

- **E(X1+X2+…) = E(X1)+E(X2)+…. = P(E1)+P(E2)+….**

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Suppose everyone (n) puts their cell phone in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back? Guess??

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

Need to calculate: $E[\sum_{i=1}^{n} X_i]$

k	0	1
$Pr(X_i=k)$	1-(1/n)	1/n

$E[X_i] = Pr(X_i = 1)$

 Why? Symmetry! All phones equally likely.

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

So,

So, we expect just **one** student to get his or her own cell phone back…

Independent of n!!

$$E[X] = E[X_1 + X_2 + \ldots + X_n]$$

$$= E[X_1] + E[X_2] + \ldots + E[X_n]$$

$$= 1/n + 1/n + \ldots + 1/n$$

$$= 1$$

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Suppose there are N couples at a party, and suppose m (random) people get "sleepy and leave"… What is the expected number of ("complete") couples left?

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

So, what do we know about X_i?

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_i] = \Pr(X_i = 1)$

How?

(# of ways of choosing m from everyone else) /(# of ways of choosing m from all)

$$= \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$$

$E[X_1] + E[X_2] + \ldots + E[X_n]$

$= n \times E[X_1]$ $= (2N-m)(2N-m-1)/2(2N-1)$

4. **[10 points] Expectation**

Imagine there are N couples at a party, and suppose m people get sleepy and have to go home. When a person goes home, his or her date has to leave with them. We need to find the expected number of couples left at the party.

(a) First, let's consider a single couple, couple i. Define a random variable X_i to be:

$$X_i = \begin{cases} 1 & \text{if couple } i \text{ stays} \\ 0 & \text{if couple } i \text{ leaves} \end{cases}$$

This sort of function is known as an *indicator*.

What is the expected value of X_i? Clearly explain your answer.

Solution 1: The couple stays only if neither gets sleepy. To compute this probability, we count the number of ways to choose m sleepy people from everyone else excluding this couple, then divide by the total number of ways to choose m sleepy people. So we get:

$$p = \frac{\binom{2N-2}{m}}{\binom{2N}{m}} = \frac{\frac{(2N-2)!}{(2N-2-m)!m!}}{\frac{(2N)!}{(2N-m)!m!}} = \frac{(2N-m)(2N-m-1)}{2N(2N-1)}$$

The expected value of X_i is thus $\mathbf{E}(X_i) = p * 1 + (1-p) * 0 = p$.

Solution 2: The couple stays only if neither gets sleepy, which occurs with probability $p = (1 - \frac{m}{2N})(1 - \frac{m}{2N-1})$. This is because $\frac{m}{2N}$ is the probability that the first person in the couple gets sleepy, and $1 - \frac{m}{2N}$ is the probability that the first person in the couple does *not* get sleepy and stays.

$1 - \frac{m}{2N-1}$ is the probability that the *second* person in the couple does not get sleepy and stays, *given* that the first person does not get sleepy and stays. The denominator is only $2N - 1$ since one person is already known to not be sleepy. If you fiddle with the algebra, you'll see that this equation is equal to the one in the previous solution.

Again, the expected value of X_i is thus $\mathbf{E}(X_i) = p * 1 + (1 - p) * 0 = p$.

(b) What is the expected number of couples left at the party? Your answer should be a function of the variables N and m.

Hint: Suppose we have n random variables X_i and we wish to compute the expected value of $\sum_{i=1}^{n} X_i$. This means we have sum of n functions and we wish to find the expected value of the sum. We can do this by using the *linearity of expectation* which establishes that $\mathbf{E}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathbf{E}(X_i)$. This means that we can find the answer by computing the expectation for each X_i separately and then summing the results.

Solution 1: We use our result for **Solution 1** of part a. By linearity of expectation, the expected number of couples left at the party is $\sum_{i=1}^{N} \mathbf{E}(X_i) = \sum_{i=1}^{N} p = N * p$ which is equal to:

$$\frac{(2N - m)(2N - m - 1)}{2(2N - 1)}$$

Solution 2: We use our result for **Solution 2** of part a. By linearity of expectation, the expected number of couples left at the party is $\sum_{i=1}^{N} \mathbf{E}(X_i) = \sum_{i=1}^{N} p = N * p$ which is equal to:

$$N\left(1 - \frac{m}{2N}\right)\left(1 - \frac{m}{2N-1}\right)$$

Note: Algebraic manipulation shows that the two solutions are equivalent.