Function of Random Variables- Derived Distributions

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5. Functions of a Random Variable

Let X be a r.v defined on the model (Ω, F, P), and suppose $g(x)$ is a function of the variable x. Define

$$Y = g(X). \qquad (5\text{-}1)$$

Is Y necessarily a r.v? If so what is its PDF $F_Y(y)$, pdf $f_Y(y)$?

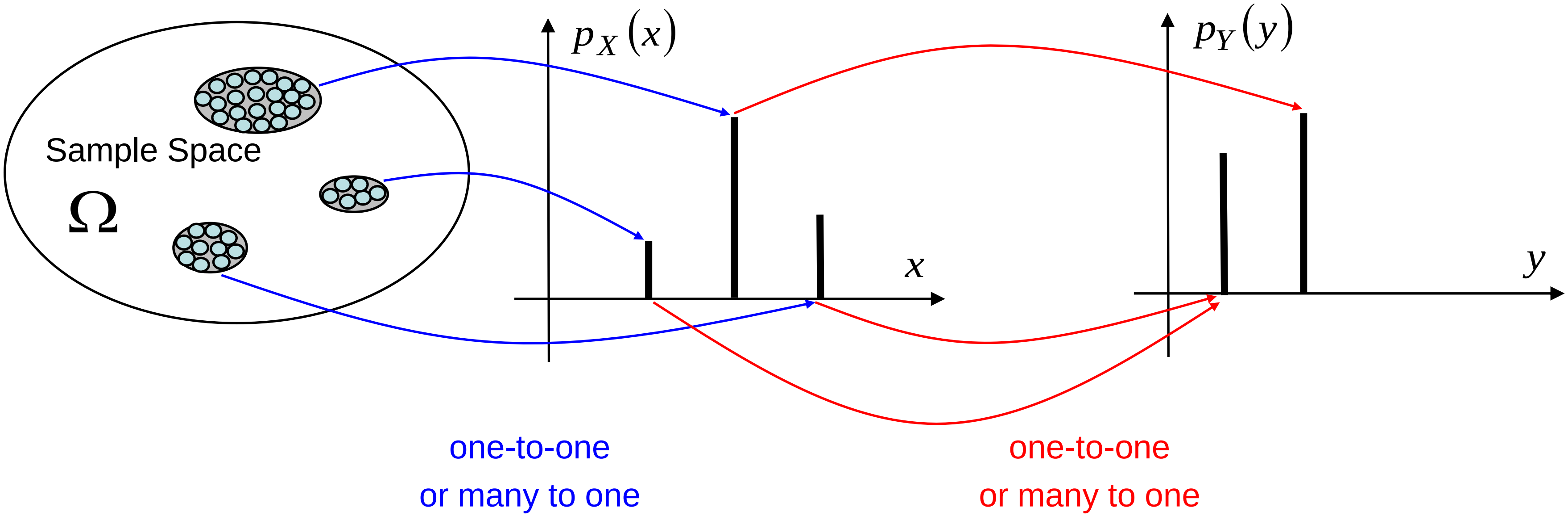
Clearly if Y is a r.v, then for every Borel set B, the set of for which must belong to F. Given that X is a r.v, this is assured if is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if X is a r.v, so is Y, and for every Borel set B

(5-2)

Functions of Random Variables (1/2)

- Given a random variable X, other random variables can be generated by applying various transformations on X
  - Linear $Y = g(X) = aX + b$

    Daily temperature in degree Fahrenheit (Y) — Daily temperature in degree Celsius (X)
  - Nonlinear $Y = g(X) = \log X$



Functions of Random Variables (2/2)

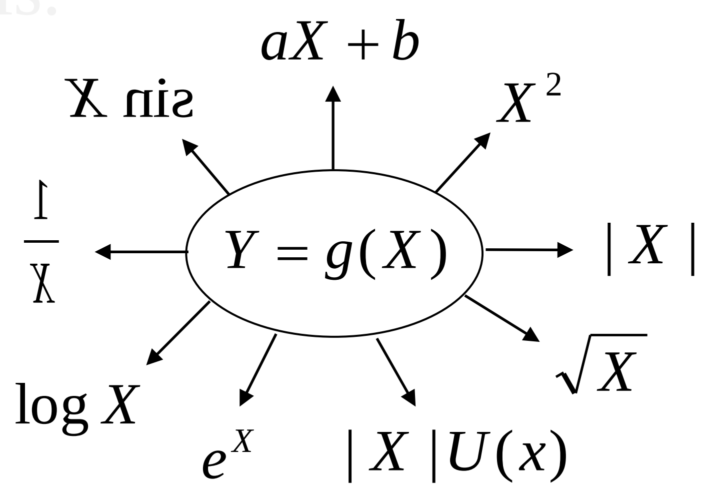
- That is, if Y is an function of X ($Y = g(X)$), then Y is also a random variable
  - If X is discrete with PMF $p_X(x)$, then Y is also discrete and its PMF can be calculated using

$$p_Y(y) = \sum_{\{x | g(x) = y\}} p_X(x)$$

In particular

$$F_Y(y) = P(Y(\xi) \le y) = P\big(g(X(\xi)) \le y\big) = P\big(X(\xi) \le g^{-1}(-\infty, y]\big). \quad \text{(5-3)}$$

Thus the distribution function as well of the density function of *Y* can be determined in terms of that of *X*. To obtain the distribution function of *Y*, we must determine the Borel set on the *x*-axis such that for every given *y*, and the probability of that set. At this point, we shall consider some of the following functions to illustrate the technical details.



Functions of Random Variables: An Example

$$p_Y(y) = \sum_{\{x | g(x) = y\}} p_X(x)$$

Example 2.1. Let $Y = |X|$ and let us apply the preceding formula for the PMF p_Y to the case where

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4], \\ 0 & \text{otherwise.} \end{cases}$$

The possible values of Y are $y = 0, 1, 2, 3, 4$. To compute $p_Y(y)$ for some given value y from this range, we must add $p_X(x)$ over all values x such that $|x| = y$. In particular, there is only one value of X that corresponds to $y = 0$, namely $x = 0$. Thus,

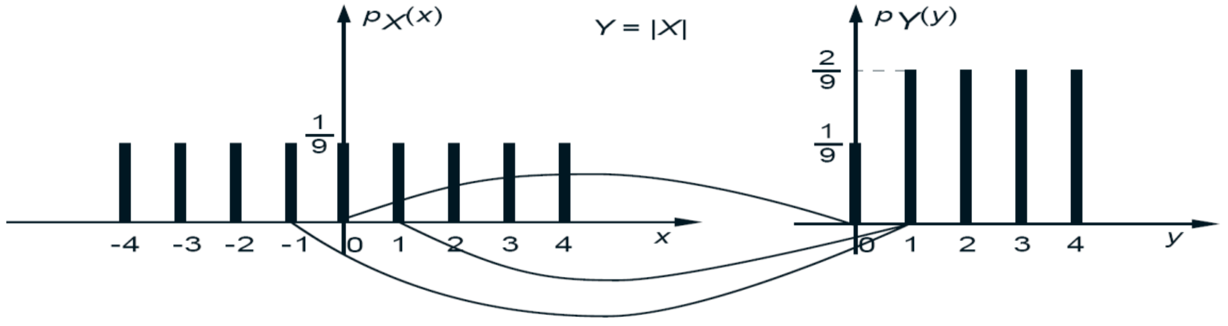
$$p_Y(0) = p_X(0) = \frac{1}{9}.$$

Also, there are two values of X that correspond to each $y = 1, 2, 3, 4$, so for example,

$$p_Y(1) = p_X(-1) + p_X(1) = \frac{2}{9}.$$

Thus, the PMF of Y is

$$p_Y(y) = \begin{cases} 2/9 & \text{if } y = 1, 2, 3, 4, \\ 1/9 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$



LOTUS: *Law of the unconscious statistician*

Special case: Calculate Expectation of g(X)

Theorem: Suppose that g(X) is a function of a random variable X, & the probability mass function of X is $p_x(x)$. Then the expected value of g(X) is

$$E[g(X)] = \sum_{x} g(x)\, p_x(x)$$

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)	y	p(y)
-2	0.1		
-1	0.2		
1	0.3		
2	0.4		

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)
1	0.5

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)
1	0.5
4	0.5

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)	yp(y)
1	0.5	0.5
4	0.5	2.0

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. ***the definition of expected value***, &
2. the previous theorem.

x	p(x)
-2	0.1
-1	0.2
1	0.3
2	0.4

y	p(y)	yp(y)
1	0.5	0.5
4	0.5	2.0
	E(Y) =	2.5

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using
1. the definition of expected value, &
2. ***the previous theorem***.

x	p(x)	y
-2	0.1	4
-1	0.2	1
1	0.3	1
2	0.4	4

Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. ***the previous theorem***.

x	p(x)	y	$yp_X(x)$
-2	0.1	4	0.4
-1	0.2	1	0.2
1	0.3	1	0.3
2	0.4	4	1.6

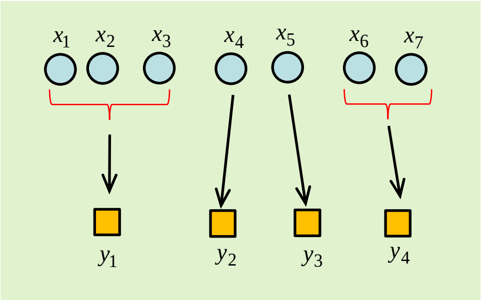
Example: Suppose $Y = X^2$ & the distribution of X is as given below. Determine the mean of g(X) by using

1. the definition of expected value, &
2. ***the previous theorem***.

x	$p(x)$	y	$yp_X(x)$
-2	0.1	4	0.4
-1	0.2	1	0.2
1	0.3	1	0.3
2	0.4	4	1.6
		E(Y) =	2.5

Expectations for Functions of Random Variables

- Let X be a random variable with PMF p_X, and let $g(X)$ be a function of X. Then, the expected value of the random variable $g(X)$ is given by



$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

- To verify the above rule
  - Let $Y = g(X)$, and therefore $p_Y(y) = \sum_{\{x \mid g(x)=y\}} p_X(x)$

$$\mathbf{E}[g(X)] = \mathbf{E}[Y] = \sum_y y p_Y(y)$$

$$= \sum_y y \sum_{\{x \mid g(x)=y\}} p_X(x) = \sum_y \sum_{\{x \mid g(x)=y\}} g(x) p_X(x)$$

$$= \sum_x g(x) p_X(x)$$

?

An Example

- **Example 2.3:** For the random variable X with PMF

$$p_X(x) = \begin{cases} 1/9, & \text{if x is an integer in the range [-4, 4],} \\ 0, & \text{otherwise} \end{cases}$$

Discrete Uniform Random Variable

$$\mathbf{E}[X] = \sum_x x p_X(x) = \frac{1}{9} \sum_{x=-4}^{4} x = 0$$

$$\text{var}(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \sum_x (x - \mathbf{E}[X])^2 p_X(x) = \frac{1}{9} \sum_{x=-4}^{4} x^2 = \frac{60}{9}$$

$$\text{Or, let } Z = (X - \mathbf{E}[X])^2 = X^2$$

$$\Rightarrow p_Z(\text{z}) = \begin{cases} 2/9, & \text{if } z = 1,4,9,16 \\ 1/9, & \text{if } z = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{var}(X) = \mathbf{E}[Z] = \sum_z z p_Z(\text{z}) = \frac{60}{9}$$

General case:

Now, we would like to find the distribution of $Y=g(X)$

Method 1

Example 5.1: $Y = aX + b$ (5-4)

Solution: Suppose $a > 0$.

$$F_Y(y) = P(Y(\xi) \le y) = P(aX(\xi) + b \le y) = P\left(X(\xi) \le \frac{y-b}{a} \right) = F_X\left(\frac{y-b}{a} \right). \tag{5-5}$$

and

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a} \right). \tag{5-6}$$

On the other hand if $a < 0$, then

$$F_Y(y) = P(Y(\xi) \le y) = P(aX(\xi) + b \le y) = P\left(X(\xi) > \frac{y-b}{a} \right)$$

$$= 1 - F_X\left(\frac{y-b}{a} \right), \tag{5-7}$$

and hence

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a} \right). \tag{5-8}$$

From (5-6) and (5-8), we obtain (for all a)

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \tag{5-9}$$

Example 5.2: $Y = X^2.$ (5-10)

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\Big(X^2(\xi) \le y\Big). \tag{5-11}$$

If $y < 0,$ then the event $\left\{ X^2(\xi) \le y \right\} = \phi$, and hence

$$F_Y(y) = 0, \quad y < 0. \tag{5-12}$$

For $y > 0,$ from Fig. 5.1, the event $\{Y(\xi) \le y\} = \{X^2(\xi) \le y\}$ is equivalent to $\{x_1 < X(\xi) \le x_2\}.$

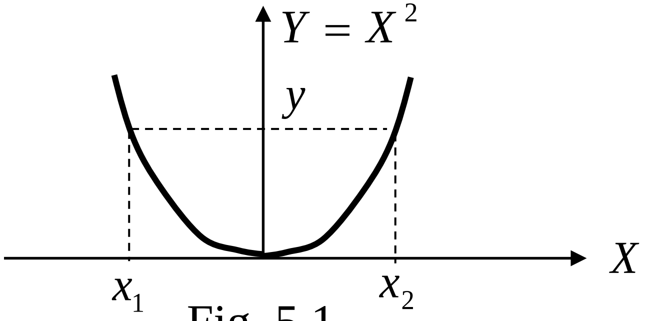


Fig. 5.1

Hence

$$F_Y(y) = P\big(x_1 < X(\xi) \le x_2\big) = F_X(x_2) - F_X(x_1)$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0. \qquad (5\text{-}13)$$

By direct differentiation, we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\Big(f_X(\sqrt{y}) + f_X(-\sqrt{y})\Big), & y > 0, \\ 0, & \text{otherwise.} \end{cases} \qquad (5\text{-}14)$$

If $f_X(x)$ represents an even function, then (5-14) reduces to

$$f_Y(y) = \frac{1}{\sqrt{y}}\, f_X\big(\sqrt{y}\big)\, U(y). \qquad (5\text{-}15)$$

In particular if $X \sim N(0,1)$, so that

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \qquad (5\text{-}16)$$

and substituting this into (5-14) or (5-15), we obtain the p.d.f of $Y = X^2$ to be

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y). \tag{5-17}$$

On comparing this with (3-36), we notice that (5-17) represents a Chi-square r.v with *n* = 1, since $\Gamma(1/2) = \sqrt{\pi}.$ Thus, if *X* is a Gaussian r.v with $\mu = 0,$ then $Y = X^2$ represents a Chi-square r.v with one degree of freedom (*n* = 1).

https://www.statlect.com/probability-distributions/chi-square-distribution

Method 2

Note: As a general approach, given $Y = g(X)$, first sketch the graph $y = g(x)$, and determine the range space of *y*. Suppose $a < y < b$ is the range space of $y = g(x)$. Then clearly for $y < a$, $F_Y(y) = 0$, and for $y > b$, $F_Y(y) = 1$, so that $F_Y(y)$ can be nonzero only in $a < y < b$. Next, determine whether there are discontinuities in the range space of *y*. If so evaluate $P\big(Y(\xi) = y_i\big)$ at these discontinuities. In the continuous region of *y*, use the basic approach

$$F_Y(y) = P\big(g(X(\xi)) \le y\big)$$

and determine appropriate events in terms of the r.v *X* for every *y*. Finally, we must have $F_Y(y)$ for $-\infty < y < +\infty$, and obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} \quad \text{in} \quad a < y < b.$$

However, if $Y = g(X)$ is a continuous function, it is easy to establish a direct procedure to obtain $f_Y(y)$. A continuos function $g(x)$ with $g'(x)$ nonzero at all but a finite number of points, has only a finite number of maxima and minima, and it eventually becomes monotonic as $|x| \to \infty$. Consider a specific y on the y-axis, and a positive increment Δy as shown in Fig. 5.4

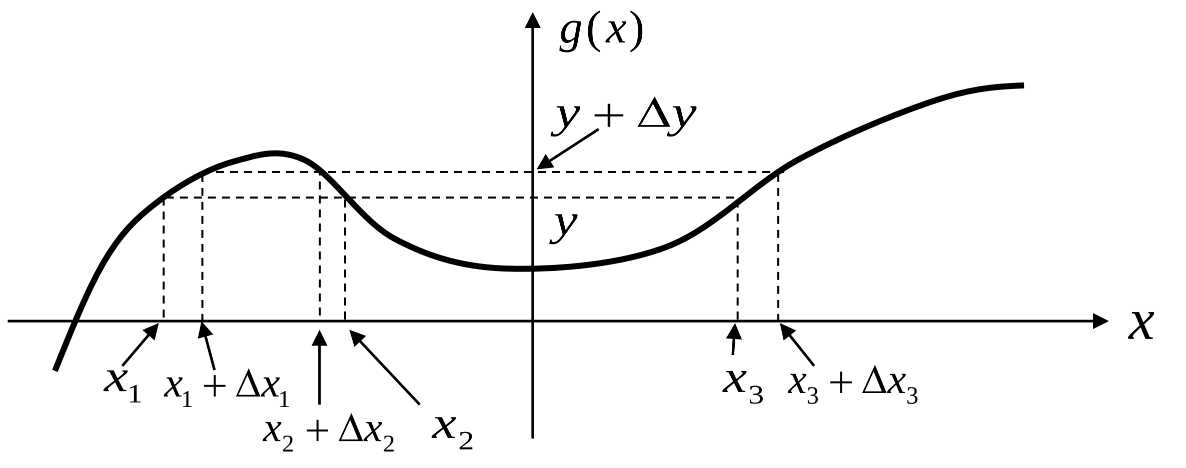


Fig. 5.4

$f_Y(y)$ for $Y = g(X)$, where $g(\cdot)$ is of continuous type.

Using (3-28) we can write

$$P\{y < Y(\xi) \le y + \Delta y\} = \int_y^{y+\Delta y} f_Y(u)du \approx f_Y(y) \cdot \Delta y. \qquad (5\text{-}26)$$

But the event $\{y < Y(\xi) \le y + \Delta y\}$ can be expressed in terms of $X(\xi)$ as well. To see this, referring back to Fig. 5.4, we notice that the equation $y = g(x)$ has three solutions x_1, x_2, x_3 (for the specific *y* chosen there). As a result when $\{y < Y(\xi) \le y + \Delta y\}$, the r.v *X* could be in any one of the three mutually exclusive intervals

$$\{x_1 < X(\xi) \le x_1 + \Delta x_1\},\ \{x_2 + \Delta x_2 < X(\xi) \le x_2\}\ \text{ or }\ \{x_3 < X(\xi) \le x_3 + \Delta x_3\}.$$

Hence the probability of the event in (5-26) is the sum of the probability of the above three events, i.e.,

$$P\{y < Y(\xi) \le y + \Delta y\} = P\{x_1 < X(\xi) \le x_1 + \Delta x_1\}$$
$$+ P\{x_2 + \Delta x_2 < X(\xi) \le x_2\} + P\{x_3 < X(\xi) \le x_3 + \Delta x_3\}. \qquad (5\text{-}27)$$

For small $\Delta y, \Delta x_i$, making use of the approximation in (5-26), we get

$$f_Y(y)\Delta y = f_X(x_1)\Delta x_1 + f_X(x_2)(-\Delta x_2) + f_X(x_3)\Delta x_3. \qquad (5\text{-}28)$$

In this case, $\Delta x_1 > 0, \ \Delta x_2 < 0$ and $\Delta x_3 > 0,$ so that (5-28) can be rewritten as

$$f_Y(y) = \sum_i f_X(x_i)\frac{|\Delta x_i|}{\Delta y} = \sum_i \frac{1}{|\Delta y / \Delta x_i|} f_X(x_i) \qquad (5\text{-}29)$$

and as $\Delta y \to 0,$ (5-29) can be expressed as

$$f_Y(y) = \sum_i \frac{1}{|dy/dx|_{x_i}} f_X(x_i) = \sum_i \frac{1}{|g'(x_i)|} f_X(x_i). \qquad (5\text{-}30)$$

The summation index *i* in (5-30) depends on *y*, and for every *y* the equation $y = g(x_i)$ must be solved to obtain the total number of solutions at every *y*, and the actual solutions $x_1, x_2, \cdots$ all in terms of *y*.

Examples

For example, if $Y = X^2$, then for all $y > 0$, $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$ represent the two solutions for each *y*. Notice that the solutions x_i are all in terms of *y* so that the right side of (5-30) is only a function of *y*. Referring back to the example $Y = X^2$ (Example 5.2) here for each $y > 0$, there are two solutions given by $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$. ($f_Y(y) = 0$ for $y < 0$). Moreover

$$\frac{dy}{dx} = 2x \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x = x_i} = 2\sqrt{y}$$

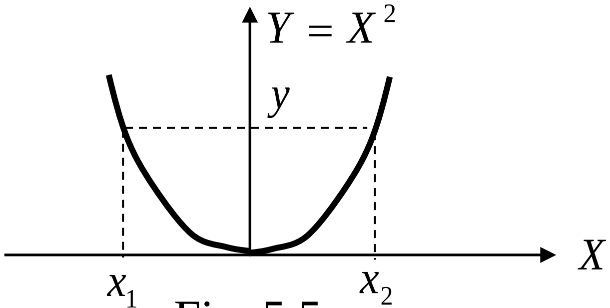


Fig. 5.5

and using (5-30) we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right), & y > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (5\text{-}31)$$

which agrees with (5-14).

Example 5.5: $Y = \dfrac{1}{X}.$ Find $f_Y(y).$ (5-32)

Solution: Here for every *y*, $x_1 = 1/y$ is the only solution, and

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x=x_1} = \frac{1}{1/y^2} = y^2,$$

and substituting this into (5-30), we obtain

$$f_Y(y) = \frac{1}{y^2}\, f_X\!\left(\frac{1}{y}\right). \tag{5-33}$$

In particular, suppose *X* is a Cauchy r.v as in (3-39) with parameter α so that

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}, \quad -\infty < x < +\infty. \tag{5-34}$$

In that case from (5-33), $Y = 1/X$ has the p.d.f

$$f_Y(y) = \frac{1}{y^2}\frac{\alpha/\pi}{\alpha^2 + (1/y)^2} = \frac{(1/\alpha)/\pi}{(1/\alpha)^2 + y^2}, \quad -\infty < y < +\infty. \tag{5-35}$$

But (5-35) represents the p.d.f of a Cauchy r.v with parameter $1/\alpha$. Thus if $X \sim C(\alpha)$, then $1/X \sim C(1/\alpha)$.

Example 5.6: Suppose $f_X(x) = 2x/\pi^2, \;\; 0 < x < \pi$, and $Y = \sin X$. Determine $f_Y(y)$.

Solution: Since *X* has zero probability of falling outside the interval $(0, \pi)$, $y = \sin x$ has zero probability of falling outside the interval $(0,1)$. Clearly $f_Y(y) = 0$ outside this interval. For any $0 < y < 1$, from Fig.5.6(b), the equation $y = \sin x$ has an infinite number of solutions $\cdots, x_1, x_2, x_3, \cdots,$ where $x_1 = \sin^{-1} y$ is the principal solution. Moreover, using the symmetry we also get $x_2 = \pi - x_1$ etc. Further,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

so that

$$\left| \frac{dy}{dx} \right|_{x = x_i} = \sqrt{1 - y^2}.$$

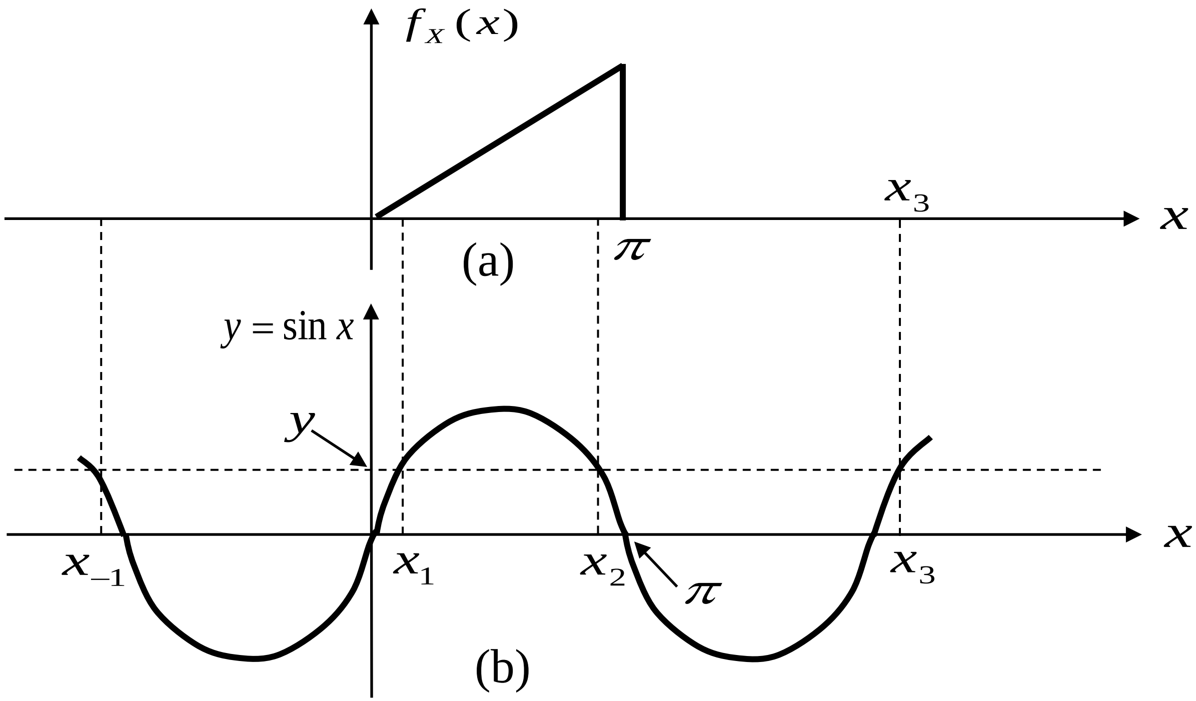


Fig. 5.6

Using this in (5-30), we obtain for $0 < y < 1$,

$$f_Y(y) = \sum_{\substack{i=-\infty \\ i \neq 0}}^{+\infty} \frac{1}{\sqrt{1-y^2}} f_X(x_i). \tag{5-36}$$

But from Fig. 5.6(a), in this case $f_X(x_{-1}) = f_X(x_3) = f_X(x_4) = \cdots = 0$ (Except for $f_X(x_1)$ and $f_X(x_2)$ the rest are all zeros).

Thus (Fig. 5.7)

$$f_Y(y) = \frac{1}{\sqrt{1-y^2}}\left(f_X(x_1) + f_X(x_2)\right) = \frac{1}{\sqrt{1-y^2}}\left(\frac{2x_1}{\pi^2} + \frac{2x_2}{\pi^2}\right)$$

$$= \frac{2(x_1 + \pi - x_1)}{\pi^2\sqrt{1-y^2}} = \begin{cases} \dfrac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (5\text{-}37)$$

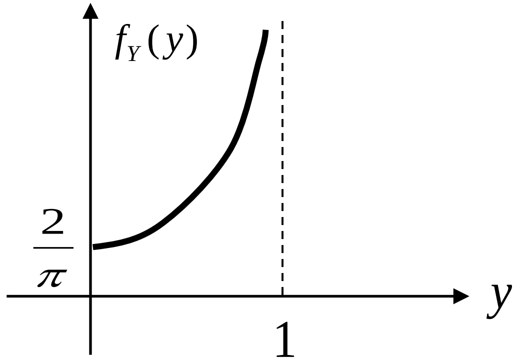


Fig. 5.7

Example 5.7: Let $Y = \tan X$ where $X \sim U(-\pi/2, \pi/2)$. Determine $f_Y(y)$.

Solution: As x moves from $(-\pi/2,\ \pi/2)$, y moves from $(-\infty, +\infty)$. From Fig.5.8(b), the function $Y = \tan X$ is one-to-one for $-\pi/2 < x < \pi/2$. For any y, $x_1 = \tan^{-1} y$ is the principal solution. Further

$$\frac{dy}{dx} = \frac{d\tan x}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

https://www.desmos.com/calculator

so that using (5-30)

$$f_Y(y) = \frac{1}{|dy/dx|_{x=x_1}} f_X(x_1) = \frac{1/\pi}{1+y^2}, \quad -\infty < y < +\infty, \qquad (5\text{-}38)$$

which represents a Cauchy density function with parameter equal to unity (Fig. 5.9).

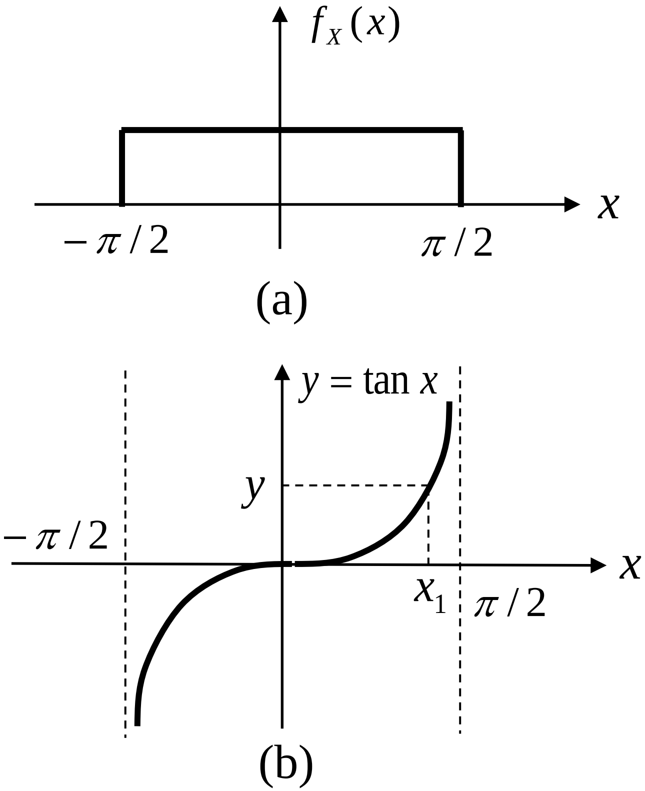


Fig. 5.8

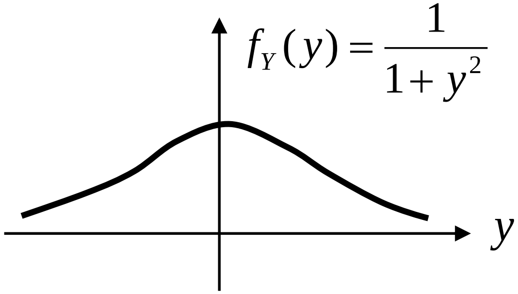


Fig. 5.9

Functions of a discrete-type r.v

Suppose X is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \cdots, x_i, \cdots \tag{5-39}$$

and $Y = g(X)$. Clearly Y is also of discrete-type, and when $x = x_i, \;\; y_i = g(x_i)$, and for those y_i

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \cdots, y_i, \cdots \tag{5-40}$$

Example 5.8: Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,2,\cdots \tag{5-41}$$

Define $Y = X^2 + 1$. Find the p.m.f of Y.
Solution: X takes the values $0,1,2,\cdots,k,\cdots$ so that Y only takes the value $1, 2, 5, \cdots, k^2 + 1, \cdots$ and

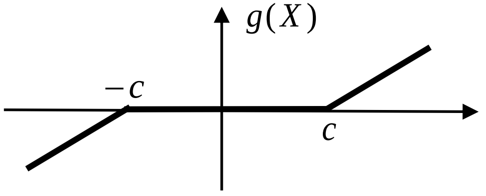
$$P(Y = k^2 + 1) = P(X = k)$$

so that for $j = k^2 + 1$

$$P(Y = j) = P\left(X = \sqrt{j-1}\right) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \cdots, k^2 + 1, \cdots. \qquad (5\text{-}42)$$

Example 5.3: Let

$$Y = g(X) = \begin{cases} X - c, & X > c, \\ 0, & -c < X \le c, \\ X + c, & X \le -c. \end{cases}$$



(a)

In this case

$$P(Y = 0) = P(-c < X(\xi) \le c) = F_X(c) - F_X(-c). \qquad (5\text{-}18)$$

For $y > 0$, we have $x > c$, and $Y(\xi) = X(\xi) - c$ so that

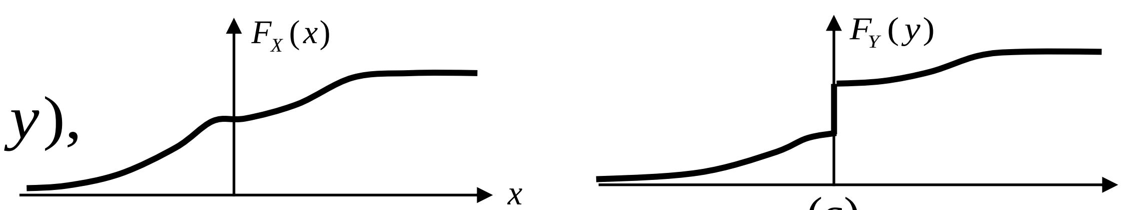
$$\begin{aligned} F_Y(y) &= P\big(Y(\xi) \le y\big) = P(X(\xi) - c \le y) \\ &= P\big(X(\xi) \le y + c\big) = F_X(y + c), \quad y > 0. \end{aligned} \qquad (5\text{-}19)$$

Similarly $y < 0$, if $x < -c$, and $Y(\xi) = X(\xi) + c$ so that

$$\begin{aligned} F_Y(y) &= P\big(Y(\xi) \le y\big) = P(X(\xi) + c \le y) \\ &= P\big(X(\xi) \le y - c\big) = F_X(y - c), \quad y < 0. \end{aligned} \qquad (5\text{-}20)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y + c), & y > 0, \\ [F_X(c) - F_X(-c)]\delta(y), & \\ f_X(y - c), & y < 0. \end{cases}$$



(b) (c)

Fig. 5.2

Example 5.4: Half-wave rectifier

$$Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0. \end{cases} \qquad (5\text{-}22)$$

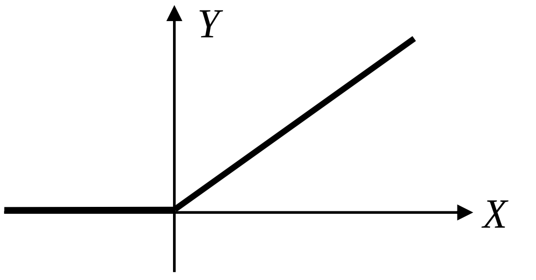


Fig. 5.3

In this case

$$P(Y = 0) = P(X(\xi) \le 0) = F_X(0). \qquad (5\text{-}23)$$

and for $y > 0$, since $Y = X$,

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(X(\xi) \le y\big) = F_X(y). \qquad (5\text{-}24)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y), & y > 0, \\ F_X(0)\delta(y) & y = 0, \\ 0, & y < 0, \end{cases} = f_X(y)U(y) + F_X(0)\delta(y). \qquad (5\text{-}25)$$

(۲)

کتاب احتمال Ross ویرایش نهم ص 62 :

$$Y_1 = g_1(X_1, X_2)$$

$$Y_2 = g_2(X_1, X_2)$$

و g_1, g_2 دو شرط زیر را برآورده کنند. ① معادلات $\begin{cases} y_1 = g_1(x_1, x_2) \\ y_2 = g_2(x_1, x_2) \end{cases}$ را بتوان برای x_1 و x_2 بر حسب y_1 و y_2 حل کرد

و جوابها را نیز $\begin{cases} x_1 = h_1(y_1, y_2) \\ x_2 = h_2(y_1, y_2) \end{cases}$ را بتوان به دست آورد.

② g_1 و g_2 در همه نقاط (x_1, x_2) دارای مشتق جزئی باشند و در نتیجه برای همه نقاط داشته باشیم:

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \neq 0$$

می‌توان نشان داد که چگالی توأم (Y_1, Y_2) به صورت زیر است

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)\,|J(x_1, x_2)|^{-1}$$

که $\begin{cases} x_1 = h_1(\ ) \\ x_2 = h_2(\ ) \end{cases}$ که h_1 و h_2 نظیر معکوس g_1 و g_2 هستند

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اثبات: مثل کتاب را ثابت کنید.

$$P(Y_1 \le y_1, Y_2 \le y_2) = \iint_{\substack{(x_1, x_2):\\ g_1(x_1,x_2) \le y_1 \\ g_2(x_1,x_2) \le y_2}} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2$$

در اسلاید ص 63 در آن گفته

مثال: X و Y دو م.ت مستقل و با توزیع گاما با پارامترهای (α, λ) و (β, λ) هستند، اثبات کنید $U = X+Y$ و $V = \frac{X}{X+Y}$ را هم حساب کنید.

$$f_{X,Y}(x,y) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \times \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)} = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda(x+y)} x^{\alpha-1} y^{\beta-1}$$

if $g_1(x,y) = x+y$, $g_2(x,y) = \frac{x}{x+y}$ → then:

$$\begin{cases} \frac{\partial g_1}{\partial x} = \frac{\partial g_1}{\partial y} = 1 \\ \frac{\partial g_2}{\partial x} = \frac{y}{(x+y)^2}, \quad \frac{\partial g_2}{\partial y} = -\frac{x}{(x+y)^2} \end{cases}$$

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y}$$

و حل: x و y را از ترتیب u و v حساب کنیم داریم:

$$x = uv, \quad y = u(1-v)$$

$x + y = uv + u(1-v) = u$

$$f_{U,V}(u,v) = f_{X,Y}[uv, u(1-v)]\,u = \frac{\lambda e^{-\lambda u} \lambda^{\alpha+\beta-1} u^{\alpha+\beta-1} v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} \times e^{-\lambda u} (uv)^{\alpha-1} [u(1-v)]^{\beta-1} \times u$$

$$= \underbrace{\frac{\lambda e^{-\lambda u}(\lambda u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}}_{①} \times \underbrace{\frac{\Gamma(\alpha+\beta)\, v^{\alpha-1}(1-v)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}}_{②}$$

① و ② مستقل

① توزیع گاما با پارامتر $(\alpha+\beta, \lambda)$

② توزیع بتا با پارامتر (α, β)