Laplace and Z Transforms

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OUTLINE:

- Z-transform:
  - Definition;
  - Properties;
  - Inversion.
- Laplace transform:
  - Definition;
  - Properties;
  - Inversion.

1. Why transforms

Why we are going to consider them separately:

- most problems for those who did not take specific math courses;
- provide a way to analyze queuing systems.

Types of the transforms:

- Z transform;
- Laplace transform;
- Fourier transform;
- ...

How we call transforms:

- just transform (referring to any transform);
- Z-transform: (probability) generating function;
- Laplace transform: Laplace-Stieltjes transform.

Why we are going to use transforms:

- they naturally appear in analysis of queues;
- they simplify the calculation;
- sometimes they are the only tool.

What kind of transforms we are going to consider:

- Z transform for discrete RVs.
- Laplace transform for continuous RVs;

We basically follow:

- L. Kleinrock, " Queuing systems, Volume I: Theory," John Wiley & Sons;
- R. Gabel, R. Roberts, " Signals and linear systems," John Wiley & Sons;
- Internet, e.g. www.wikipedia.org

2. Z transform

Assume: we are given discrete function defined on RV X , which takes nonnegative values , X ∈{0, 1, 2, . . .}.

Denote the point probabilities by p_i $$p_i = P\{X = i\} \quad (1)$$

What we want: compress it into a single one such that:

- it passes unchanged through the system;
- we can decompress it.

For a (complex valued) sequence $(a_n)_{n\in\mathbb{N}}$ there is the associated generating func

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and the z-Transform

$$Z(a)(z) = \sum_{n=0}^{\infty} a_n z^{-n}$$

Do the following:

- tag each value in sequence multiplying by z^i:
  - why z^i: i is unique, thus, z^i is unique for each p_i.
- get a single function depending on z only $G(z)$ (or $G_X(z)$; also $X(z)$ or $\hat{X}(z)$) by summing all terms:

$$G(z) = G_X(z) = \sum_{i=0}^{\infty} p_i z^i = E[z^X] \quad (2)$$

  - which is called z-transform (or generating function or geometric transform).

2. Z transform

Rationale

- A handy way to record all the values $\{p_0, p_1, ...\}$; z is a 'bookkeeping variable'
- Often G(z) can be explicitly calculated (a simple analytical expression)
- When G(z) is given, one can conversely deduce the values $\{p_0, p_1, ...\}$
- Some operations on distributions correspond to much simpler operations on the generating functions
- Often simplifies the solution of recursive equations, eg. Using generating function to find an explicit formula for Fibonacci no. F_n

Condition of existence for z-transform:

- terms in a sequence grow no faster than geometrically;
- meaning that if there is $a > 0$ for which the following holds:

$$\lim_{i \to \infty} \frac{|p_i|}{a^i} = 0 \quad (3)$$

  – for this sequence z-transform is unique.

Analyticity:

- the sum of all terms in p_i must be finite;
    - if so, then $G(z)$ is analytic on a unit circle $|z| \leq 1$;
- in this case we have:

$$G(1) = \sum_{i=0}^{\infty} p_i \quad (4)$$

Note: analyticity means that the function has unique derivative.
In mathematics, an analytic function is a function that is locally given by a convergent power series.

1. Getting z-transforms: Kleinrock page 328

Delta function $\delta_i = 1, i = 0, \delta_i = 0, i \neq 0$**:**

$$G(z) = \sum_{i=0}^{\infty} p_i z^i$$

- since the only one term is non-zero corresponding to $i = 0$ we have from (2) ,

exactly one term in infinite summation is nonzero and so we have transform pair:

$$G(z) = \sum_{i=0}^{\infty} \delta_i z^i \qquad \delta_i \leftrightarrow z^0 = 1 \tag{5}$$

Delta function shifted to the right by k**:** $\delta_{i-k} = 1, i = k, \delta_i = 0, i \neq k$**:**

- since the only one term is non-zero corresponding to $i = k$ we have:

$$\delta_{i-k} \leftrightarrow z^k \tag{6}$$

Unit step function: $u_i = 1, i = 0, 1, \ldots$ **:**

- recall that $u_i = 0$ for $i < 0$;
- we have geometric series:

$$u_i \leftrightarrow \sum_{i=0}^{\infty} 1 z^i = \frac{1}{1-z} \tag{7}$$

Geometric series: $p_i = A\alpha^i$, $i = 0, 1, \ldots$:

- calculate z-transform as follows:

$$G(z) = \sum_{i=0}^{\infty} A\alpha^i z^i = A\sum_{i=0}^{\infty}(\alpha z)^i = \frac{A}{1-\alpha z} \tag{8}$$

- therefore, we have:

$$p_i = A\alpha^i \leftrightarrow \frac{A}{1-\alpha z} \tag{9}$$

- z-transform is analytic for $|z| \leq 1/\alpha$.

Arbitrary sequence: $\{ p_0 = -2, p_1 = 0, p_2 = 4, p_3 = -6 \}$:

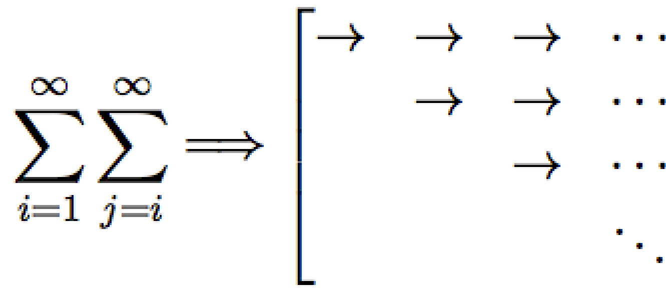
- calculate z-transform as follows:

$$G(z) = \sum_{i=0}^{3} p_i z^i = -2 + 4z^2 - 6z^3 \tag{10}$$

SEQUENCE		z-TRANSFORM
1. $f_n \quad n = 0, 1, 2, \ldots$	$\Leftrightarrow$	$F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$		1
3. δ_{n-k}		z^k
4. $u_n = 1 \quad n = 0, 1, 2, \ldots$		$\frac{1}{1 - z}$
5. u_{n-k}		$\frac{z^k}{1 - z}$
6. $A\alpha^n$		$\frac{A}{1 - \alpha z}$
7. $n\alpha^n$		$\frac{\alpha z}{(1 - \alpha z)^2}$

SEQUENCE	z-TRANSFORM
8. n	$\frac{z}{(1-z)^2}$
9. $n^2\alpha^n$	$\frac{\alpha z(1+\alpha z)}{(1-\alpha z)^3}$
10. n^2	$\frac{z(1+z)}{(1-z)^3}$
11. $(n+1)\alpha^n$	$\frac{1}{(1-\alpha z)^2}$
12. $(n+1)$	$\frac{1}{(1-z)^2}$
13. $\frac{1}{m!}(n+m)(n+m-1)\cdots(n+1)\alpha^n$	$\frac{1}{(1-\alpha z)^{m+1}}$
14. $\frac{1}{n!}$	e^z

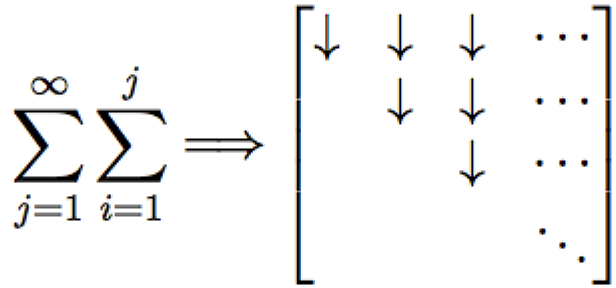
2. Properties of z-transform



Convolution property: Let X and Y be *independent* RVs with corresponding distributions:

$p_i = P\{X = i\} > 0 \quad i = 0, 1, \ldots;$

$q_j = P\{Y = j\} > 0 \quad j = 0, 1, \ldots;$



- denote their transforms by $G_X(z)$ and $G_Y(z)$;
- convolution is defined as follows: $p_i \odot q_i \leftrightarrow \sum_{k=0}^{i} p_{i-k} q_k$ (11)
  - derive the transform of the convolution as:

$$p_i \odot q_i \leftrightarrow \sum_{i=0}^{\infty} (p_i \odot q_i) z^i = \sum_{i=0}^{\infty} \sum_{k=0}^{i} p_{i-k} q_k z^{i-k} z^k \qquad (12)$$

  - change the summation $\sum_{i=0}^{\infty} \sum_{k=0}^{i} = \sum_{k=0}^{\infty} \sum_{i=k}^{\infty}$ to get as:

$$p_i \odot q_i \leftrightarrow \sum_{k=0}^{\infty} q_k z^k \sum_{i=k}^{\infty} p_{i-k} z^{i-k} = \sum_{k=0}^{\infty} q_k z^k \sum_{m=0}^{\infty} p_m z^m = G_X(z) G_Y(z) \qquad (13)$$

SEQUENCE			z-TRANSFORM
1. f_n	$n = 0, 1, 2, \ldots$	$\Leftrightarrow$	$F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $af_n + bg_n$			$aF(z) + bG(z)$
3. $a^n f_n$			$F(az)$
4. $f_{n/k}$	$n = 0, k, 2k, \ldots$		$F(z^k)$
5. f_{n+1}			$\frac{1}{z}[F(z) - f_0]$
6. f_{n+k}	$k > 0$		$\frac{F(z)}{z^k} - \sum_{i=1}^{k} z^{i-k-1} f_{i-1}$
7. f_{n-1}			$zF(z)$
8. f_{n-k}	$k > 0$		$z^k F(z)$
9. nf_n			$z\frac{d}{dz}F(z)$

SEQUENCE	z-TRANSFORM
10. $n(n-1)(n-2),\ldots,(n-m+1)f_n$	$z^m \frac{d^m}{dz^m} F(z)$
11. $f_n \circledast g_n$	$F(z)G(z)$
12. $f_n - f_{n-1}$	$(1-z)F(z)$
13. $\sum_{k=0}^{n} f_k \quad n = 0, 1, 2, \ldots$	$\frac{F(z)}{1-z}$
14. $\frac{\partial}{\partial a} f_n$ (a is a parameter of f_n)	$\frac{\partial}{\partial a} F(z)$
15. Series sum property	$F(1) = \sum_{n=0}^{\infty} f_n$
16. Alternating sum property	$F(-1) = \sum_{n=0}^{\infty} (-1)^n f_n$
17. Initial value theorem	$F(0) = f_0$
18. Intermediate value theorem	$\frac{1}{n!} \frac{d^n F(z)}{dz^n}\Big\|_{z=0} = f_n$
19. Final value theorem	$\lim_{z \to 1} (1-z)F(z) = f_\infty$

3. Inverting z-transform **Why we need it:**

- sometimes we need to get p_i when we have $G(z)$;
- example: queuing systems, we will see...

Methods to invert transforms: three methods

1- Develop G(z) in a power series, from which the p_i can be identified as the coefficients of the z^i. The coefficients can also be calculated by derivation (this is actually uses intermediate value theorem (property 18):

$$p_i = \frac{1}{i!}\frac{d^i G(z)}{dz^i}\bigg|_{z=0} = \frac{1}{i!}G^{(i)}(0) \qquad (14)$$

– complicated when many terms are required. Example: see next slide

2- By inspection: decompose G(z) in parts the inverse transforms of which are known; e.g. the partial fractions (usage of the inversion formula (see, for example, Kleinrock, " Queuing systems, Vol. I"))

3. By a (path) integral on the complex plane

$$p_i = \frac{1}{2\pi j}\oint \frac{G(z)}{z^{i+1}}\, dz$$

path encircling the origin (must be chosen so that the poles of G(z) are outside the path)

Note: all methods are, at least, time-consuming!!!

$$p_X^T(z) = p_X(0) + p_X(1)z + p_X(2)z^2 + \ldots + p_X(k)z^k + \ldots$$

which yields through successive differentiation

$$p_X(k) = \frac{1}{k!}\left[\frac{d^k}{dz^k}p_X^T(z)\right]_{z=0} \qquad k = 0, 1, 2, \ldots$$

More Examples:

$$G(z) = \frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots$$

$$\Rightarrow \qquad p_i = \begin{cases} 1 & for\ i\ even \\ 0 & for\ i\ odd \end{cases}$$

Example 2

$$G(z) = \frac{1}{(1-z)(2-z)} = \frac{2}{1-z} - \frac{2}{2-z} = \frac{2}{1-z} - \frac{1}{1-z/2}$$

Since $\frac{A}{1-az}$ corresponds to sequence $A \cdot a^i$ we deduce

$$p_i = 2\,.\,(1)^i - 1\,.\left(\frac{1}{2}\right)^i = 2 - \left(\frac{1}{2}\right)^i$$

4. Example: inverting using inspection method

Basis: partial-fraction expansion:

- technique for expressing a rational function of z as a sum of simple terms;
- the idea: get elements that are easily invertible;
- possible when $G(z)$ is rational function of z: $G(z) = N(z)/D(z)$;
- possible when the degree of nominator is less than that of denominator (if not, make it so!).

What we want:

- get terms like:

$$A\alpha^i \leftrightarrow \frac{A}{1-\alpha z}, \quad \frac{1}{m!}(i+m)(i+m-1)\dots(i+1)\alpha^i \leftrightarrow \frac{1}{(1-\alpha z)^{m+1}} \tag{15}$$

What we then use:

- sum of the transforms equals to the transform of the sum:

$$ap_i + bq_i = aGX(z) + bG_Y(z) \tag{16}$$

Assumptions:

- $D(z)$ in $G(z) = N(z)/D(z)$ is already in factored form:

$$D(z) = \prod_{l=1}^{k}(1 - \alpha_l z)^{m_l} \tag{17}$$

  - lth root is at $1/\alpha_l$ occurring m_l times.

- **Note:** putting $D(z)$ in the factored form can be complicated.

If above is satisfied you may get $F(z)$ in the following form:

In the general form below: lth root is at $1/\alpha_l$ occurring m_l times

$$G(z) = \frac{A_{11}}{(1-\alpha_1 z)^{m_1}} + \frac{A_{12}}{(1-\alpha_1 z)^{m_1-1}} + \cdots + \frac{A_{1m_1}}{(1-\alpha_1 z)} + \frac{A_{21}}{(1-\alpha_2 z)^{m_2}} + \frac{A_{22}}{(1-\alpha_2 z)^{m_2-1}} + \cdots$$
$$+ \frac{A_{2m_2}}{(1-\alpha_2 z)} + \cdots + \frac{A_{k1}}{(1-\alpha_k z)^{m_k}} + \frac{A_{k2}}{(1-\alpha_k z)^{m_k-1}} + \cdots + \frac{A_{km_k}}{(1-\alpha_k z)} \qquad (18)$$

where coefficients are given by

$$A_{lj} = \frac{1}{(j-1)!}\left(-\frac{1}{\alpha_l}\right)^{j-1} \frac{d^{j-1}}{dz^{j-1}}\left((1-\alpha_l z)^{m_l}\frac{N(z)}{D(z)}\right)\Bigg|_{z=1/\alpha_l} \qquad (19)$$

Multiplying by $(1-\alpha_l z)^{m_l}$ discards multi- root z= $1/\alpha_l$ in denominator and thus the expression at z= $1/\alpha_l$ is unambiguous

Example: Kleinrock page 336

$$G(z) = \left(\frac{4z^2(1-8z)}{(1-4z)(1-2z)^2}\right) \tag{20}$$

Do the following:

- observe that denominator and nominator have the same degree (i.e. 3);
  - we have to put it in a proper form (degree of nominator must be strictly less);
  - to do so factor out two powers of z^2 to get:

$$G(z) = z^2\left(\frac{4(1-8z)}{(1-4z)(1-2z)^2}\right) \tag{21}$$

- denote the rest by $R(z)$:

$$R(z) = \frac{4(1-8z)}{(1-4z)(1-2z)^2} \tag{22}$$

  - there are three poles of denominator: single pole $z = 1/4$ and double pole $z = 1/2$;
  - we have $k = 2$, $\alpha_1 = 4$, $m_1 = 1$, $\alpha_2 = 2$, $m_2 = 2$.

- now we can rewrite $R(z) = [4(1-8z)]/[(1-4z)(1-2z)^2]$ as

$$R(z) = \frac{4(1-8z)}{(1-4z)(1-2z)^2} = \frac{A_{11}}{1-4z} + \frac{A_{21}}{(1-2z)^2} + \frac{A_{22}}{1-2z} \quad (23)$$

- get elements A_{11}, A_{21} and A_{22} as follows:

$$A_{11} = (1-4z)R(z)\Big|_{z=\frac{1}{4}} = \frac{4\left(1-\left(\frac{8}{4}\right)\right)}{\left(1-\left(\frac{2}{4}\right)\right)^2} = -16$$

$$A_{21} = (1-2z)^2 R(z)\Big|_{z=\frac{1}{2}} = \frac{4\left(1-\left(\frac{8}{2}\right)\right)}{(1-(4/2))} = 12$$

(24)

$$A_{22} = -\frac{1}{2}\frac{d}{dz}(1-2z)^2 R(z)\Big|_{z=\frac{1}{2}} = -\frac{1}{2}\frac{d}{dz}\frac{4(1-8z)}{(1-4z)}\Big|_{z=\frac{1}{2}} = -\frac{1}{2}\frac{(1-4z)(-32)-4(1-8z)(-4)}{(1-4z)^2}\Big|_{z=\frac{1}{2}} = 8$$

(25)

- we get the following expression for $R(z)$:

$$R(z) = -\frac{16}{1-4z} + \frac{12}{(1-2z)^2} + \frac{8}{1-2z}$$

- check that you got the same as initially had (place terms under common denominator);
- now we can invert $R(z)$ by inspection:
  - first and third terms are in the form: $A\,\alpha^i \Leftrightarrow A/(1-\alpha z)$;

$$-\frac{16}{1-4z} \Leftrightarrow -16(4)^i$$

$$\frac{8}{1-2z} \Leftrightarrow 8(2)^i$$

  - second term is in the form: $(1/m!)(i+m)(i+m-1)\ldots(i+1)\,\alpha^i \Leftrightarrow 1/(1-\alpha z)^{m+1}$;

$$\frac{12}{(1-2z)^2} \Leftrightarrow 12(i+1)(2)^i$$

– using the linearity $ag_i + bq_i = aG_X(z) + bG_Y(z)$ we get:

$$R(z) \leftrightarrow q_i = \begin{cases} 0 & i < 0 \\ -16(4)^i + 12(i+1)(2)^i + 8(2)^i & i = 0,1,\ldots \end{cases} \tag{26}$$

– using property 8 we take into account factor z^2 in $G(z)$:

$$p_i = -16(4)^{i-2} + 12(i-1)(2)^{i-2} + 8(2)^{i-2} \quad i = 2,3,\ldots \tag{27}$$

– finally optimizing the expression we have for p_i:

$$\begin{aligned} p_i &= 0, \qquad i < 2, \\ p_i &= (3i-1)(2)^i - (4)^i \qquad i = 2,3,\ldots \end{aligned} \tag{28}$$

Notes: other examples are in detail in R. Gabel, R. Roberts, "Signals and linear systems".

3. The Laplace transform: Kleinrock page 338

Assume: we are given continuous function $f(t)$ defined on nonzero values:

$$f(t) = 0, t < 0 \tag{29}$$

What we want: compress it into a single one such that:

- it passes unchanged through the system;
- we can decompress it.

Do the following:

- tag each value of $f(t)$ multiplying by e^{-st}:
  - why e^{-st}: t is unique, thus, e^{-st} is unique for each $f(t)$;
  - why e^{-st}: exponentials pass through linear time-invariant systems unchanged.
- get a single function by integrating over all non-zero values:

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{30}$$

  - which gives **two-sided** Laplace transform.

Our case: since $f(t)$ defined on nonzero values we have:

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{31}$$

- which gives **one-sided** Laplace transform (0 means 0^- which means $0 - \epsilon$ for $\epsilon > 0$, $\epsilon \to 0$).

Condition of existence for Laplace transform:

- terms in a sequence must grow no faster than exponential;
- meaning that if there is real number σ_a for which the following holds:

$$\lim_{\tau\to\infty} \int_0^\tau |f(t)|e^{\sigma_a t}dt < 0 \tag{32}$$

  – Laplace transform exists and unique.

Analyticity of the Laplace transform:

- the integral of $f(t)$ must be finite;
- if so, then $F(s)$ is analytic on a right hand plane of $Re(s) \geq 0$:

$$F(0) = \int_0^\infty f(t)dt \tag{33}$$

3.1. Getting Laplace transform

Example: one sided exponential function:

$$f(t) = \begin{cases} Ae^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{34}$$

- get the Laplace transform as follows

$$f(t) \leftrightarrow F(s) = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(a+s)t}\,dt = \frac{A}{s+a} \quad . \tag{35}$$

Example: unit step function:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{36}$$

- consider it as a special case of one-sided exponential function to get:

$$u(t) \leftrightarrow F(s) = \frac{1}{s} \tag{37}$$

FUNCTION		TRANSFORM
1. $f(t) \quad t \geq 0$	$\Leftrightarrow$	$F^*(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$
2. $\delta(t)$ (unit impulse)		1
3. $\delta(t-a)$		e^{-as}
4. $u_n(t) \triangleq \frac{d}{dt} u_{n-1}(t)$		s^n
5. $u_{-1}(t) \triangleq u(t)$ (unit step)		$\frac{1}{s}$

FUNCTION	TRANSFORM
6. $u_{-1}(t-a)$	$\frac{e^{-as}}{s}$
7. $u_{-n}(t) \triangleq \frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$
8. $Ae^{-at}u(t)$	$\frac{A}{s+a}$
9. $te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
10. $\frac{t^n}{n!}e^{-at}u(t)$	$\frac{1}{(s+a)^{n+1}}$

2. Properties of the Laplace transform

Convolution property:

- consider $f(t) > 0$, $g(t) > 0$ for $t \geq 0$ only;
- denote their transforms by $F(s)$ and $G(s)$;
- convolution is defined as follows:

$$f(t) \odot g(t) \leftrightarrow \int_{-\infty}^{\infty} f(t-x)g(x)dx \tag{38}$$

  – in our case the lower limit is 0^-, the upper limit is ∞.

- derive the transform of the convolution as: $\int_{t=0}^{\infty}\int_{x=0}^{t} = \int_{x=0}^{\infty}\int_{t=x}^{\infty}$

$$f(t) \odot g(t) \leftrightarrow \int_{t=0}^{\infty} (f(t) \odot g(t))e^{-st}dt = \int_{t=0}^{\infty}\int_{x=0}^{t} f(t-x)g(x)dx\, e^{-st}dt$$

$$= \int_{t=0}^{\infty}\int_{x=0}^{t} f(t-x)e^{-s(t-x)}dt\, g(x)e^{-sx}dx = \int_{x=0}^{\infty}\int_{t=x}^{\infty} f(t-x)e^{-s(t-x)}dt\, g(x)e^{-sx}dx \tag{39}$$

$$= \int_{x=0}^{\infty} g(x)e^{-sx}dx \int_{v=0}^{\infty} f(v)e^{-sv}dv = F(s)G(s)$$

FUNCTION		TRANSFORM
1. $f(t) \quad t \geq 0$	$\Leftrightarrow$	$F^*(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$
2. $af(t) + bg(t)$		$aF^*(s) + bG^*(s)$
3. $f\left(\frac{t}{a}\right) \quad (a > 0)$		$aF^*(as)$
4. $f(t-a)$		$e^{-as}F^*(s)$
5. $e^{-at}f(t)$		$F^*(s+a)$
6. $tf(t)$		$-\frac{dF^*(s)}{ds}$
7. $t^n f(t)$		$(-1)^n \frac{d^n F^*(s)}{ds^n}$
8. $\frac{f(t)}{t}$		$\int_{s_1=s}^{\infty} F^*(s_1)\,ds_1$
9. $\frac{f(t)}{t^n}$		$\int_{s_1=s}^{\infty} ds_1 \int_{s_2=s_1}^{\infty} ds_2 \cdots \int_{s_n=s_{n-1}}^{\infty} ds_n F^*(s_n)$

FUNCTION	TRANSFORM
10. $f(t) \circledast g(t)$	$F^*(s)G^*(s)$
11.† $\dfrac{df(t)}{dt}$	$sF^*(s)$
12.† $\dfrac{d^n f(t)}{dt^n}$	$s^n F^*(s)$
13.† $\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{F^*(s)}{s}$
14.† $\underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{n \text{ times}} f(t)(dt)^n$	$\dfrac{F^*(s)}{s^n}$
15. $\dfrac{\partial}{\partial a} f(t)$ [a is a parameter]	$\dfrac{\partial}{\partial a} F(s)$
16. Integral property	$F^*(0) = \int_{0^-}^{\infty} f(t)\,dt$
17. Initial value theorem	$\lim_{s\to\infty} sF^*(s) = \lim_{t\to 0} f(t)$
18. Final value theorem	$\lim_{s\to 0} sF^*(s) = \lim_{t\to\infty} f(t)$ if $sF^*(s)$ is analytic for Re $(s) \geq 0$

3.3. Two-sided Laplace transform

If $f(t)$ may the nonzero anywhere on the axis:

$$f(t) \leftrightarrow F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{40}$$

- define the following functions:

$$f_{-}(t) = \begin{cases} f(t) & t < 0 \\ 0 & t \geq 0 \end{cases}, \qquad f_{+}(t) = \begin{cases} 0 & t < 0 \\ f(t) & t \geq 0 \end{cases}. \tag{41}$$

- one may get Laplace transform as follows:

$$f(t) = f_{-}(t) + f_{+}(t) \tag{42}$$

- we have the following property:

$$F(s) = F_{-}(-s) + F_{+}(s), \quad f_{-}(t) \leftrightarrow F_{-}(-s), \quad f_{+}(t) \leftrightarrow F_{+}(s) \tag{43}$$

4. Inverting Laplace transforms

There are the following methods:

- inspection method;
- formal inversion integral method.

Inspection method:

- use partial-fraction expansion to:
  - rewrite $F(s)$ as a sum of terms;
  - each term should be recognizable as a transform pair.
- use linearity property to:
  - invert the transform term by term;
  - sum the result to recover $f(t)$.

Note: we have to ensure that $F(s)$ is a rational function of s and can be written as:

$$F(s) = N(s)/D(s) \tag{44}$$

Do the following:

- ensure that the degree of the nominator is less than that of denominator:
  - if this is not the case, make it so;
  - to do so divide $N(s)$ by $D(s)$ until the remainder is less than the degree of $D(s)$;
  - partial-fraction expansion must be carried out for remainder;
  - powers of s can be taken into account using transform 4 (see table).

- $D(s)$ in $F(s) = N(s)/D(s)$ is already in factored form:

$$D(s) = \prod_{i=1}^{k} (s + \alpha_i)^{m_i} \tag{45}$$

  - ith root is at $1/\alpha_i$ occurring m_i times.
- **note:** putting $D(s)$ in the factored form can be complicated.

If the above satisfied:

- rewrite $F(s)$ as follows:

$$F(s) = \frac{B_{11}}{(s+\alpha_1)^{m_1}} + \frac{B_{12}}{(s+\alpha_1)^{m_1-1}} + \cdots + \frac{B_{1m_1}}{(s+\alpha_1)}$$
$$+ \frac{B_{21}}{(s+\alpha_2)^{m_2}} + \frac{B_{22}}{(s+\alpha_2)^{m_2-1}} + \cdots + \frac{B_{2m_2}}{(s+\alpha_2)}$$
$$+ \cdots \tag{46}$$
$$+ \frac{B_{k1}}{(s+\alpha_k)^{m_k}} + \frac{B_{k2}}{(s+\alpha_k)^{m_k-1}} + \cdots + \frac{B_{km_k}}{(s+\alpha_k)}$$

- coefficients are given by

$$B_{ij} = \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \left((s+\alpha_i)^{mi} \frac{N(s)}{D(s)}\right|_{s=-\alpha_i} \tag{47}$$

Example:

$$F(s) = \frac{8(s^2 + 3s + 1)}{(s+3)(s+1)^3} \quad . \tag{48}$$

- the denominator is already in factored form;
- the degree of the denominator (4) is greater than that of the nominator (2);
- we have $k = 2$, $\alpha_1 = 3$, $m_1 = 1$, $\alpha_2 = 1$, $m_2 = 3$;
- we write $F(s)$ as:

$$F(s) = \frac{B_{11}}{s+3} + \frac{B_{21}}{(s+1)^3} + \frac{B_{22}}{(s+1)^2} + \frac{B_{23}}{s+1} \tag{49}$$

- it is easy to derive B_{11} and B_{21}:

$$B_{11} = (s+3)F(s)|_{s=-3} = (s+3)\frac{8(s^2+3s+1)}{(s+3)(s+1)^3}\bigg|_{s=-3} = 8\ \frac{9-9+1}{(-2)^3} = -1 \tag{50}$$

$$B_{21} = (s+1)^3 F(s)\Big|_{s=-1} = 8\ \frac{1-3+1}{2} = -4 \tag{51}$$

- derive B_{22} differentiating as follows:

$$B_{22} = \frac{d}{ds}\left.\frac{8(s^2+3s+1)}{(s+3)}\right|_{s=-1}$$
$$= \left.\frac{8(s+3)(2s+3)-(s^2+3s+1)(1)}{(s+3)^2}\right|_{s=-1}$$
$$= \left.\frac{8(s^2+6s+8)}{(s+3)^2}\right|_{s=-1} = 8\,\frac{1-6+8}{s^2} = 6 \tag{52}$$

- derive B_{23} differentiating B_{22} once more (what we had prior to evaluation at $s = -1$):

$$B_{23} = \frac{1}{2!}\frac{d^2}{ds^2}\left.\left(\frac{8(s^2+3s+1)}{(s+3)}\right)\right|_{s=-1} = \frac{1}{2}\,8\,\frac{d}{ds}\left.\left(\frac{(s^2+6s+8)}{(s+3)^2}\right)\right|_{s=-1}$$
$$= 4\left.\frac{(s+3)^2(2s+6)-(s^2+6s+8)(s+3)}{(s+3)^4}\right|_{s=-1} = 4\,\frac{2^2 4-(1-6+8)(2)(2)}{2^4} = 1 \tag{53}$$

- finally, we have the following expression for $F(s)$: (54)

$$F(s) = \frac{-1}{s+3} + \frac{-4}{(s+1)^3} + \frac{6}{(s+1)^2} + \frac{1}{s+1}$$

- finally, we have after inversion: 1st and last term using 8 and 2nd and 3rd using 10 of the Table, thus we have

$$f(t) = -e^{-3t} - 2t^2e^{-t} + 6e^{-t} + e^{-t} \quad t \geq 0 \tag{55}$$

$$f(t) = 0 \quad t < 0$$

Checking for errors when doing partial-fraction expansion:

- once we have partial-fraction expansion:
  - combine terms and compare to initial expression for $F(s)$.
- once we get $f(t)$:
  - try to get Laplace transform and compare to $F(s)$.

Notes: other examples are in detail in R. Gabel, R. Roberts, "Signals and linear systems".

Relations between different notions of transforms:

The Laplace transform is generally more useful than the moment generating function since it can be applied to functions that are not densities.[5]

The *moment generating function* is defined by

$$\mathcal{M}_X(\theta) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{\theta x} f_X(x)dx$$
$$= E\left[e^{\theta X}\right], \qquad \theta > 0, \qquad (5.44)$$

provided the integral exists. Expanding the exponential as a power series and taking the expectation implies that

$$\mathcal{M}_X(\theta) = 1 + \theta E[X] + \frac{\theta^2 E[X^2]}{2!} + \frac{\theta^3 E[X^3]}{3!} + \cdots . \qquad (5.45)$$

From (5.45) it is easy to see that

$$\mathcal{M}_X(0) = 1 \qquad (5.46)$$

and the nth moment of the random variable is given by

$$\mathcal{M}_X^{(n)}(0) = E[X^n], \qquad n = 1, 2, \ldots, \qquad (5.47)$$

and hence the name of the function is explained. From (5.40) we have that

$$\sigma_X^2 = \mathcal{M}_X^{(2)}(0) - \left(\mathcal{M}_X^{(1)}(0)\right)^2 . \qquad (5.48)$$

Relations between different notions of transforms:

It is interesting to note that if X is a discrete random variable with generating function $\mathcal{G}_X(z)$ then its moment generating function is given by

$$\mathcal{M}_X(\theta) = \sum_{x=0}^{\infty} P[X = x]\, e^{\theta x} = \mathcal{G}_X(e^{\theta}). \tag{5.50}$$

This equals the generating function of X evaluated at $z = \exp[\theta]$. Using this relationship we can state properties of probability generating functions that are analogous to those just given for moment generating functions.

Recall that from (4.24) the definition of the probability generating function for the discrete random variable X is

$$\mathcal{G}_X(z) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} P[X = i]\, z^i, \tag{5.51}$$

which can also be viewed as the expectation

$$\mathcal{G}_X(z) \stackrel{\text{def}}{=} E\left[z^X\right]. \tag{5.52}$$

Relations between different notions of transforms:

A function that is closely related to the moment generating function is the *Laplace transform*.[4] For a given real valued function $h(t), t \geq 0$, the Laplace transform, denoted by $h^*(s)$, is defined by

$$h^*(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sx} h(x) dx. \tag{5.54}$$

If $h(t), t \geq 0$, is a function corresponding to a density function of a nonnegative random variable X then it is clear from (5.44) that the Laplace transform of the function $h^*(s)$ equals the moment generating function of X evaluated at $-s$, that is,

$$h^*(s) = \mathcal{M}_X(-s). \tag{5.55}$$

To preserve the semantics associated with random variables we use the "$\mathcal{L}_X(s)$ notation" when dealing with the Laplace transform of a random variable X,

$$\mathcal{L}_X(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sx} f_X(x) dx \tag{5.56}$$

$$\begin{aligned} \mathcal{M}_X(\theta) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{\theta x} f_X(x) dx \\ &= E\left[e^{\theta X}\right], \qquad \theta > 0, \end{aligned} \qquad (5.44$$

Lecture: Laplace and Z transforms

Relations between different notions of transforms:

use the "$h^*(s)$ notation" otherwise. Notice that an interpretation of the transform for the random variable case is that it is the expectation

of $\exp[-sX]$ and thus can be written as

$$\mathcal{L}_X(s) = E\left[e^{-sX}\right].$$

Using (5.55) shows that the following relationship holds between the moment generating function of X and its Laplace transform

$$\mathcal{L}_X(s) = \mathcal{M}_X(-s).$$

A probabilistic interpretation of the Laplace transform is found in Problem 5.21. Like the theorem for moment generating functions, Theorem 5.30, if two random variables have the same Laplace transform then they are stochastically equal.

$$h^*(s) = \mathcal{M}_X(-s). \quad (5.55)$$

We write

$$f(t) \iff f^*(s)$$

to indicate transform pairs. The derivation of some elementary transforms is given in the following example:

Example 5.31 (Derivation of Some Simple Transforms) Recall that

$$n! = \int_0^\infty e^{-x} x^n \, dx.$$

By a simple conversion this implies that

$$\frac{t^{n-1}}{(n-1)!} \iff \frac{1}{s^n}, \quad n = 1, 2, \dots \,. \tag{5.57}$$

Let

$$h(t) = \lambda e^{-\lambda t};$$

then

$$h^*(s) = \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + s}, \tag{5.58}$$

and thus

$$\lambda e^{-\lambda t} \iff \frac{\lambda}{\lambda + s}.$$

Relations between different notions of transforms:

	Sequence	Generating Function
Definition	$f_i, i = 0, 1, \ldots$	$F(x) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} f_i x^i$

Relations between different notions of transforms:

The Laplace transform is generally more useful than the moment generating function since it can be applied to functions that are not densities.[5] The Laplace transform is the continuous counterpart of the ordinary generating function defined in Chapter 3. To see the relationship between these two functions assume that the function $h(t)$ takes on nonzero values for integer t, that is, that $h(t) = h_i$, for $t = i$ and $i = 0, 1, \ldots$, and let $H(x)$ be the generating function

$$H(x) = \sum_{i=0}^{\infty} h_i x^i.$$

$$h^*(s) = \mathcal{M}_X(-s). \tag{5.55}$$

$$\mathcal{M}_X(\theta) = \sum_{x=0}^{\infty} P[X = x] e^{\theta x} = \mathcal{G}_X(e^{\theta}). \tag{5.50}$$

Then it follows that

$$h^*(s) = \int_0^{\infty} e^{-sx} h(x) dx = \sum_{i=0}^{\infty} e^{-si} h_i = H(e^{-s}) \tag{5.59}$$

and thus the Laplace transform equals the ordinary generating function evaluated at e^{-s}. If h_i represents a probability density, then equations (5.59) and (5.55) show that we have simply established a different form of the relationship between moment generating functions and their corresponding probability generating functions for discrete random variables given in (5.50).

Relations between different notions of transforms:

Since Laplace transforms are equivalent to moment generating functions when the underlying function is a density, it follows that they satisfy analogous properties to those given for moment generating functions in Section 5.3. For example, the analogous moment property to (5.47) is given by

$$\mathcal{L}_X^{(n)}(0) = (-1)^n E[X^n]. \tag{5.60}$$

$$\mathcal{M}_X^{(n)}(0) = E[X^n], \quad n = 1, 2, \ldots, \tag{5.47}$$

1- Moment GF: assume RV X with cdf $F_X(x)$

$$M_X(\theta) = E\left[e^{\theta X}\right] = \int_{-\infty}^{\infty} e^{\theta X} dF_X(x)$$

or $M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sX} dF_X(x)$

- n-th moment $E[x^n] = m_X^{(n)}(0)$

2- Probability GF : assume RV X with pmf P_k

$$P_X(z) \triangleq E(z^X) = \sum_{k=0}^{\infty} P_k z^k \quad |z| \le 1$$

- Tail GF Q(z)= $\frac{1-P(z)}{1-z}$ $P_X(x)$>k

3- Laplace Transform of continuous function f(x) defined on nonzero values

$$\phi_X(s) = \int_0^\infty f_X(x) e^{-sx} dx = E[e^{-sX}]$$

- n-th moment

$$E(X^n) = (-1)^n \phi_X^{(n)}(0)$$

4- Characteristic function (Fourier –Stieltjes $F_X(x)$)

$$\phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} e^{j\omega x} dF_X(\omega) \quad -\infty < \omega < \infty$$

Relating Characteristic function / Moment Generating function

$$\phi_X(\omega) = M_X(\theta)\,|_{\theta=j\omega}$$

Relating Probability Generating function/ Moment Generating function

Using $e^{\theta} = z$ and $g_X(z) = E(z^X)$

$$g_X(e^{\theta}) = E[e^{\theta X}] = M_X(\theta)$$

Using $\theta = \ln(z)$ and $g_X(z) = E(z^X)$

$$M_X(\theta)|_{\ln(z)} = M_X(\ln z) = E(e^{X\ln z}) = E\left(e^{\ln z^X}\right) = E(z^X) = g_X(z)$$

Transforms- examples

- Also called **moment generating functions** of random variables

- The **transform** of the distribution of a random variable X is a function $M_X(s)$ of a free parameter s, defined by

$$M_X(s) = \mathbf{E}\left[e^{sX}\right]$$

  – If X is discrete

$$M_X(s) = \sum_x e^{sx} p_X(x)$$

  – If X is continuous

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

Illustrative Examples (1/5)

- **Example 4.22.** Let

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2, \\ 1/6, & \text{if } x = 3, \\ 1/3, & \text{if } x = 5. \end{cases}$$

$$\therefore M_X(s) = \mathbf{E}\left[e^{sX}\right] = \sum_x e^{sx} p_X(x)$$

$$= \frac{1}{2}e^{2s} + \frac{1}{6}e^{3s} + \frac{1}{3}e^{5s}$$

Notice that:

$$M_X(0) = \mathbf{E}\left[e^{0X}\right] = \sum_x e^{0x} p_X(x)$$

$$= \sum_x p_X(x) = 1$$

Illustrative Examples (2/5)

- **Example 4.23. The MGF Transform of a Poisson Random Variable.** Consider a Poisson random variable X with parameter λ : note: there is Z- transform as well

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

$$M_X(s) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{a^x}{x!} \qquad \left(\text{Let } a = e^s \lambda\right)$$

$$= e^{-\lambda} e^a \qquad \left(\because \text{McLaurin series} \left(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots\right) = e^a\right)$$

$$= e^{a-\lambda}$$

$$= e^{\lambda\left(e^s - 1\right)}$$

Illustrative Examples (3/5)

- **Example 4.24. The Transform of an Exponential Random Variable.** Let X be an exponential random variable with parameter λ :

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\begin{aligned} M_X(s) &= \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{(s-\lambda)x} dx \\ &= \lambda \left. \frac{e^{(s-\lambda)x}}{(s-\lambda)} \right|_0^\infty \qquad (\text{if } s-\lambda<0) \\ &= \frac{\lambda}{\lambda - s} \end{aligned}$$

Notice that:

$M_X(s)$ can be calculated only when $s < \lambda$

Illustrative Examples (4/5)

- **Example 4.25. The Transform of a Linear Function of a Random Variable.** Let $M_X(s)$ be the transform associated with a random variable X. Consider a new random variable $Y = aX + b$. We then have

$$M_Y(s) = \mathbf{E}\left[e^{s(aX+b)}\right] = e^{sb}\mathbf{E}\left[e^{saX}\right] = e^{sb}M_X(sa)$$

  - For example, if X is exponential with parameter $\lambda = 1$ and $Y = 2X + 3$, then

$$M_X(s) = \frac{\lambda}{\lambda - s} = \frac{1}{1-s}$$

$$M_Y(s) = e^{3s}M_X(2s) = e^{3s}\frac{1}{1-2s}$$

Illustrative Examples (5/5)

- **Example 4.26. The Transform of a Normal Random Variable.** Let X be normal with mean μ and variance σ^2.

We first calculate the transform of a standard normal random variable Y

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$M_Y(s) = \int_{-\infty}^{\infty} e^{sy} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= e^{s^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left[\left(y^2/2\right) - sy + \left(s^2/2\right)\right]} dy$$

$$= e^{s^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y-s)^2/2} dy$$

1

$$= e^{s^2/2}$$

Since we also know that $Y = \dfrac{X-\mu}{\sigma}$,

we can have $X = \sigma Y + \mu$

$$\therefore\ M_X(s) = e^{s\mu} M_Y(s\sigma)$$

$$= e^{s\mu} \cdot e^{s^2\sigma^2/2}$$

$$= e^{s\mu + \left(s^2\sigma^2/2\right)}$$

From Transforms to Moments (1/2)

- Given a random variable X, we have

$$M_X(s) = \mathbf{E}\left[e^{sx}\right] = \int_{-\infty}^{\infty} e^{sx} f_X(x)dx \qquad \text{(If } X \text{ is continuous)}$$

Or

$$M_X(s) = \mathbf{E}\left[e^{sx}\right] = \sum_x e^{sx} p_X(x) \qquad \text{(If } X \text{ is discrete)}$$

- When taking the derivative of the above functions with respect to s (for example, the continuous case)

$$\frac{dM_X(s)}{ds} = \frac{d\int_{-\infty}^{\infty} e^{sx} f_X(x)dx}{ds} = \int_{-\infty}^{\infty} xe^{sx} f_X(x)dx$$

  – If we evaluate it at $s = 0$, we can further have

the first moment of X

$$\left.\frac{dM_X(s)}{ds}\right|_{s=0} = \left.\int_{-\infty}^{\infty} xe^{sx} f_X(x)dx\right|_{s=0} = \int_{-\infty}^{\infty} xf_X(x)dx = \mathbf{E}[x]$$

From Transforms to Moments (2/2)

- More generally, taking the differentiation of $M_X(s)$ *n* times with respect to s will yield

$$\frac{d^n M_X(s)}{d^n s}\Big|_{s=0} = \int_{-\infty}^{\infty} x^n e^{sx} f_X(x)dx\Big|_{s=0} = \int_{-\infty}^{\infty} x^n f_X(x)dx = \mathbf{E}[x^n]$$

the *n*-th moment of X

Illustrative Examples (1/2)

- **Example 4.27.** Given a random variable X with PMF:

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2, \\ 1/6, & \text{if } x = 3, \\ 1/3, & \text{if } x = 5. \end{cases}$$

$$M_X(s) = \mathbf{E}\left[e^{sX}\right] = \sum_x e^{sx} p_X(x)$$

$$= \frac{1}{2}e^{2s} + \frac{1}{6}e^{3s} + \frac{1}{3}e^{5s}$$

$$\Rightarrow \mathbf{E}[X] = \frac{dM(s)}{ds}\Big|_{s=0}$$

$$= \frac{1}{2}\cdot 2\cdot e^{2s} + \frac{1}{6}\cdot 3\cdot e^{3s} + \frac{1}{3}\cdot 5\cdot e^{5s}\Big|_{s=0}$$

$$= 1 + \frac{3}{6} + \frac{5}{3} = \frac{19}{6}$$

$$\Rightarrow \mathbf{E}[X^2] = \frac{d^2M(s)}{d^2s}\Big|_{s=0}$$

$$= \frac{1}{2}\cdot 4\cdot e^{2s} + \frac{1}{6}\cdot 9\cdot e^{3s} + \frac{1}{3}\cdot 25\cdot e^{5s}\Big|_{s=0}$$

$$= 2 + \frac{9}{6} + \frac{25}{3} = \frac{71}{6}$$

Illustrative Examples (2/2)

- **Example.** Given an exponential random variable X with PMF:

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

$$M_X(s) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(s-\lambda)x} dx$$

$$= \lambda \frac{e^{(s-\lambda)x}}{(s-\lambda)} \Big|_0^\infty \qquad (\text{if } s - \lambda < 0)$$

$$= \frac{\lambda}{\lambda - s}$$

$$\Rightarrow \mathbf{E}[X] = \frac{dM_X(s)}{ds}\Big|_{s=0}$$

$$= \frac{\lambda}{(\lambda - s)^2}\Big|_{s=0}$$

$$= \frac{1}{\lambda}$$

$$\Rightarrow \mathbf{E}[X^2] = \frac{d^2 M_X(s)}{d^2 s}\Big|_{s=0}$$

$$= \frac{2\lambda}{(\lambda - s)^3}\Big|_{s=0}$$

$$= \frac{2}{\lambda^2}$$