Classification of Random Processes

There are several different ways to Classify Random Processes:

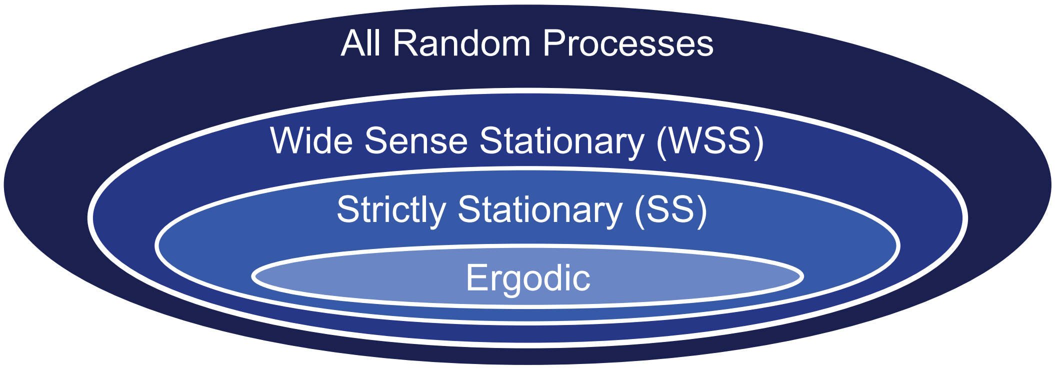
1) Type of Time Index

* Continuous–Time: x(t) $t \in (-\infty,\infty)$
* Discrete-Time: x[k] $k \in$ integers

2) Type of Values

* Continuous-Value: x(t) takes values over an interval, possibly $(-\infty,\infty)$
* Discrete-Value: x(t) takes values from a discrete set (e.g. the integers)

3) Type of Time Dependence of PDFs



$$\text{Ergodic} \subset \text{SS} \subset \text{WSS} \subset \text{All RPs}$$

"Nonstationary RP" = One that is <u>not</u> WSS

> When discussing a RP's PDF above we have allowed for the most general time dependence. However, in practice many RP's have Restricted Time-Dependence (e.g., WSS).

(Strictly) Stationary Processes

Rough Definition : A RP whose entire statistical characterization doesn't change with time

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To Get A More Precise Definition….

First consider the n^{th} order PDF and re-write it as :

$$p(x_1, x_2, \ldots x_n; t_1, t_2, \ldots, t_n)$$

$$= p(x_1, x_2, \ldots x_n; t_1, (t_1 + \Delta_2), \ldots, (t_1 + \Delta_n))$$

Depends on $\underbrace{t_1,}_{\text{Absolute Time}}\ \underbrace{\Delta_2, \ldots \Delta_n}_{\text{Relative Times}}$

Absolute Time — Relative Times

(Strictly) Stationary Processes

Precise Definition :

A Process is (strictly) stationary if, for *all* orders of *n*,

$$p(x_1,x_2,\ldots,x_n;\ t_1,\ t_1+\Delta_2,\ldots,t_1+\Delta_n)$$

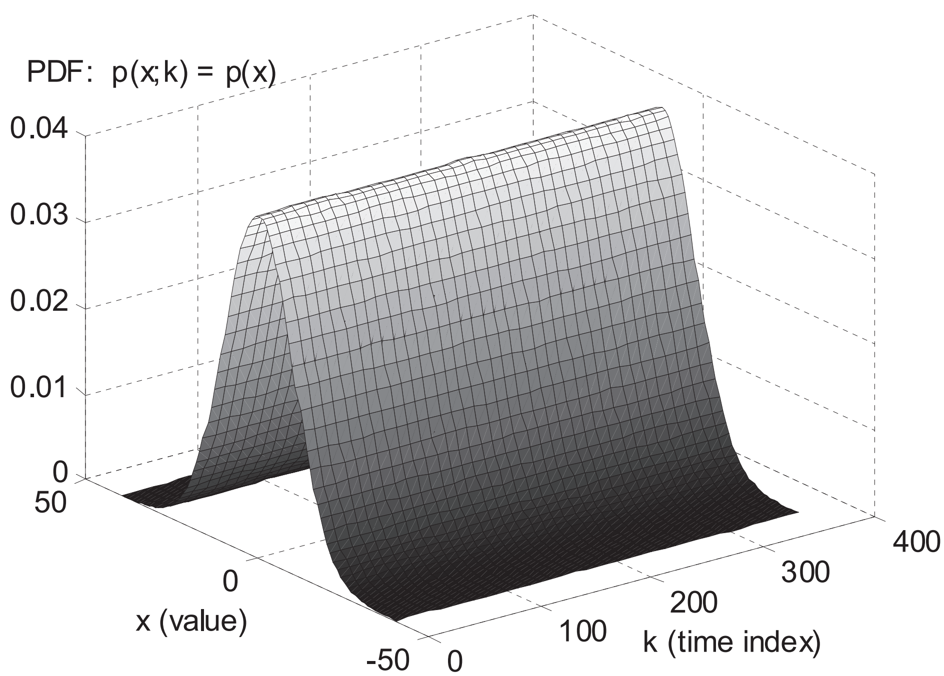
does not depend on t_1 but only on $\Delta_2,\ldots\Delta_n$

i.e. it depends *only* on relative time between points

NOTE 1: Since the 1st Order PDF $p(x_1;t_1)$ does not depend on any relative time, a SS process must have a time-independent 1st Order PDF:

$$p(x_1;t_1) = p(x_1)$$

Time-Invariant 1st Order PDF



Stationary Processes

Thus a Stationary process must have:

Constant Mean :

$$\int_{-\infty}^{\infty} x \cdot \underbrace{p(x;t)}_{= p(x) \text{ for stationary process}} dx = m_x = \text{constant}$$

Constant Variance :

$$\int_{-\infty}^{\infty} (x - m_x(t))^2 \cdot p(x;t)dx$$

$$= \int_{-\infty}^{\infty} (x - m_x)^2 \cdot p(x)dx = \sigma_x^2 = \text{constant}$$

<<These are “necessary” but not “sufficient” conditions for SS>>

Stationary Process

Note 2 : A Stationary process’s 2nd Order PDF depends only on the difference t = $t_2 - t_1$: $P(x_1,x_2; \tau)$

Thus a stationary process must have :
Autocorrelation Function that depends only on $\tau = t_2 - t_1$

$$R_x(t_1,t_2) = R_x(t_1, t_1 + \tau)$$

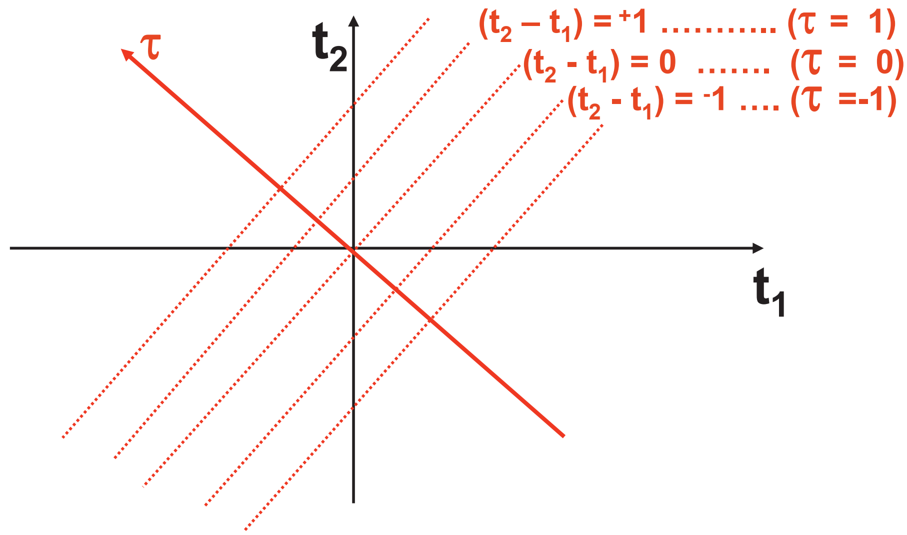
$$= \underbrace{E\{ x(t_1)\, x(t_1 + \tau)\}}_{\text{Does not depend on } t_1 \text{ if } x(t) \text{ is stationary}}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_2 \cdot x_1 \cdot p_x(x_1, x_2; \tau) dx_1 dx_2$$

$$= R_x(\tau)$$

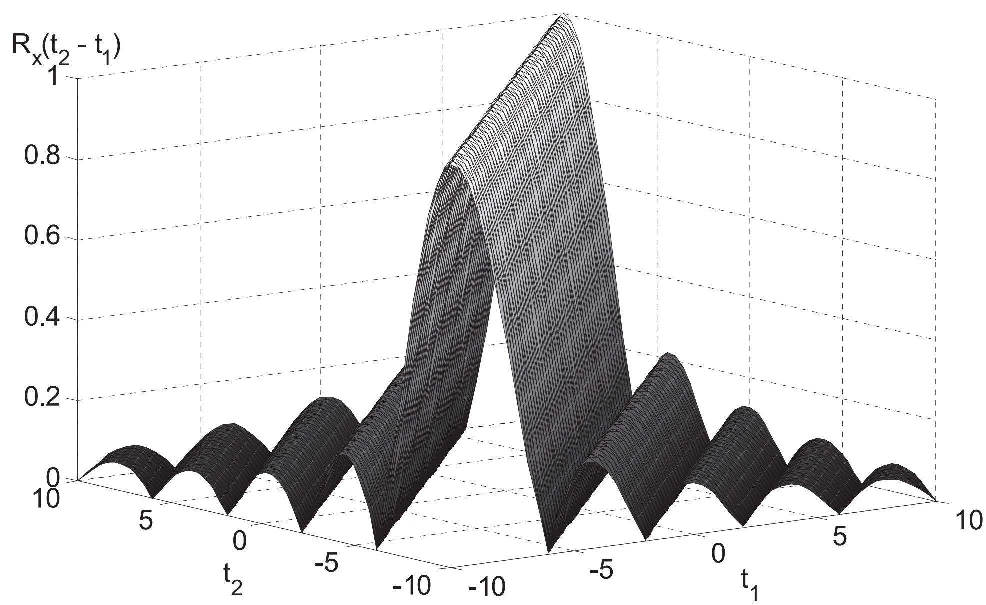
Stationary Process

Note : As a 2-D function, $R_x(t_1 - t_2)$ for a stationary process looks like this:



$R_x(t_1 - t_2)$ is constant on each of these lines → **Only a Function of $\tau = t_2 - t_1$**

An ACF That Depends Only On $t_2 - t_1$



Stationary Process

Now, A stationary process must have these 3 properties BUT … must also have the similar properties **for all** the higher Order PDF's!

That's a lot to ask of a process in practice!

In Practice we **"lower our standards"** and we are mostly interested in so-called **"wide-sense stationary" (WSS)** processes.

Wide-Sense Stationary

A Process X(t) is said to be **wide-sense stationary (WSS)** if both of the following conditions are satisfied:

1) $E\{X(t)\} = \text{Constant}$ → Constant Mean

2) $R_X(t_1, t_2) = R_X(\tau)$ → Where $\tau = t_1 - t_2$

★ ACF depends only on a time Difference

Wide-Sense Stationary

NOTE : A WSS Process has a constant Variance

Proof **By Definition** :

$$\sigma_x^2 = E\{(x(t) - \bar{x})^2\}$$

$$= E\{x^2(t) - 2\bar{x}x(t) + \bar{x}^2\}$$

$$= \underbrace{E\{x^2(t)\}}_{R_X(0)} \underbrace{- 2\bar{x}\underbrace{E\{x(t)\}}_{\bar{x}} + \bar{x}^2}_{-\bar{x}^2}$$

Wide-sense Stationary

Thus we have proved:

$$\sigma_x^2 = R_x(0) - \bar{x}^2$$

True for Stationary or WSS process

Passing Comment:

Alot of the analysis DSP engineers do centers around Specifying an Appropriate Model for a random signal expected to be encountered and Determining if the Model is WSS.

Example of RP Model

$$X(t) = A\cos(\omega_C t + \theta)$$

Not Random (A, ω_C) — Random Variable (θ)

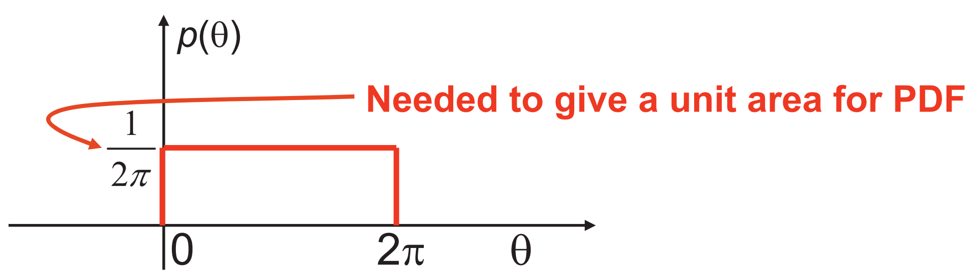
Good model for a received sinusoid since we have **no idea** what Phase the transmitter used

…thus phase could be anything ⇒ each value is equally likely

So… Model θ as a RV **uniformly distributed** between 0 & 2π

Example of RP Model (cont.)

PDF of θ : $p(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$



Q : What does this Model Say ?

Example of RP Model (cont.)

A : Transmitter (Tx) randomly "picks" a single phase value from 0 to 2π and generates a realization

$A\ COS\ (\omega_C t + \theta)$

Note : Once picked, θ doesn't change with time

Each time the Tx is turned on we <u>randomly</u> get a new phase

Example of RP Model (cont.)

Q: Which of our two "view points" is easier to <u>think</u> of for this example?

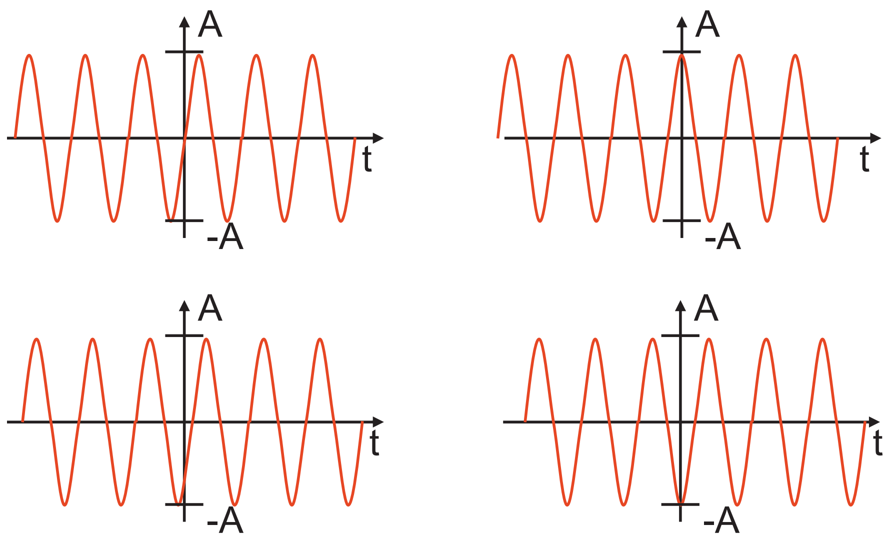
1. Sequence of RVs???
2. Collection of Waveforms & Pick One???

A : Clearly it is easier to view <u>this</u> random process as a collection of waveforms from which you randomly pick one

Remember!!! – Both Views are Still Correct

Example of RP Model (cont.)

Here are **4 realizations (sample functions)** of the ensemble of this process



Each one has a <u>different</u> Phase

Which signal you get is randomly chosen according to the PDF of Phase

Example of RP Model (cont.)

Looking at any <u>one</u> sample function doesn't give the appearance of being a random process !

<u>**BUT IT *IS* RANDOM**</u>**!** You don't know ahead of time which you were going to get.

In <u>this case</u>, randomness is best viewed as "not knowing which of the infinite possible sample functions you will get"

So… now we <u>have a model</u> for a practical signal scenario. Now What???

<u>Do analysis to characterize the model</u> !!!!

Example of RP Model (cont.)

Task : Find the **mean** and the **ACF** of this process & Ask: **Is It WSS?**

Mean $= E\{x(t)\} = E\{A\cos(\omega_c t + \theta)\}$

$= A\, E\{\cos(\omega_c t + \theta)\}$

$$= A\int_0^{2\pi} \cos(\omega_c t + \theta)\frac{1}{2\pi}\,d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega_c t + \theta)\,d\theta$$

Integrating over one full cycle for each fixed t value: ➔ = 0

Exp. Val. of *Function* of a RV
If Z is a RV w/ PDF P(z) then W = f(Z) is a new RV w/

$$E\{W\} = E\{f(Z)\} = \int f(z)P_z(z)dz$$

So Far We've Shown:
MEAN = CONSTANT!

Example of RP Model (cont.)

Auto-Correlation Function (ACF)

Until you know that the process is at least WSS you must start with the general form of $R_X(t_1,t_2)$. Then work to see if you can reduce it to $R_X(\tau)$ Form

$$R_x(t_1,t_2) = E\{x(t_1)\; x(t_2)\}$$

Plug in x(t); pull out deterministic A^2

$$= A^2\, E\{\cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)\}$$

Trig. identity

$$= \frac{A^2}{2}\left[E\{\cos(\omega_c(t_2 - t_1))\} + E\{\cos(\omega_c(t_2 + t_1) + 2\theta)\}\right]$$

$=\cos(\omega_c(t_2-t_1))$ — Nothing random inside E{.} !

=0 — Similar to analysis for mean

Example of RP Model (cont.)

$$R_x(t_1, t_2) = \frac{A^2}{2}\cos[\omega_c(t_2 - t_1)]$$

Depends only on $\tau = t_2 - t_1$

$$R_x(\tau) = \frac{A^2}{2}\cos[\omega_c \tau]$$

Have shown that this process is WSS, i.e.

Mean = constant

ACF = function of τ only

Example of RP Model (cont.)

Now… What is variance for this example?

Variance of Sinusoid w/ Random Phase is:

$$\sigma_x^2 = R_x(0) - \bar{x}^2 = \frac{A^2}{2} - 0$$

$$= \frac{A^2}{2}$$

Classic Result
Worth Remembering!!!