Birth-Death Process

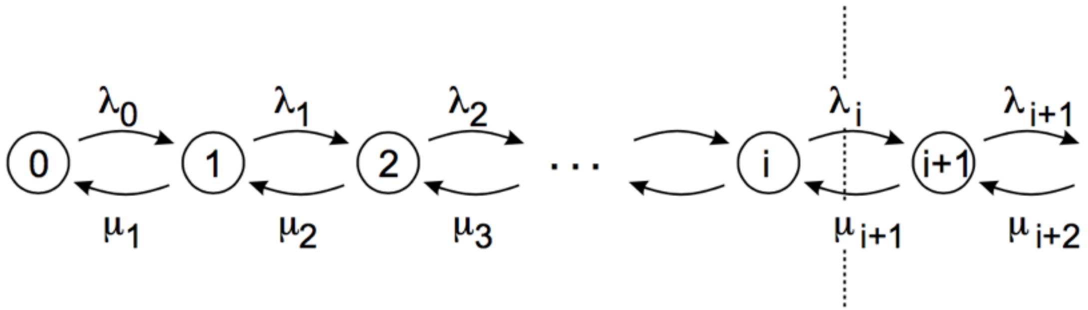
University of Tehran
Prepared by: ahmad Khonsari
Excerpt from : Virtamo Slides

- Birth-death processes

- General

- A birth-death (BD process) process refers to a Markov process with

- \- a discrete state space
  \- the states of which can be enumerated with index $i = 0,1,2,\ldots$ such that
  \- state transitions can occur only between neighboring states, $i \to i+1$ or $i \to i-1$



Transition rates

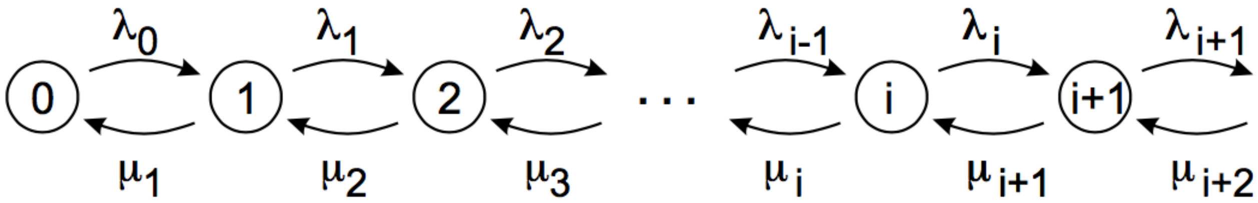
$$q_{i,j} = \begin{cases} \lambda_i & when\ j = i+1 \mid probability\ of\ birth\ in\ interval\ \Delta t\ on\ \lambda_i \Delta t \\ \mu_i & if\ j = i-1 \mid probabiltiy\ of\ death\ in\ interval\ \Delta t\ on\ \mu_i \Delta t \\ 0 & otherwise\ \mid \qquad when\ the\ system\ is\ in\ state\ i \end{cases}$$

- The equilibrium probabilities of a BD process
- We use the method of a cut = global balance condition applied on the set of states 0, 1, . . . , k. In equilibrium the probability flows across the cut are balanced (net flow =0)
- $\lambda_k \pi_k = \mu_{k+1} \pi_{k+1} \quad , k = 0,1,2, \ldots$
- We obtain the recursion
- $\pi_{k+1} = \frac{\lambda_k}{\mu_{k+1}} \pi_k$
- By means of the recursion, all the state probabilities can be expressed in terms of that of the state 0, π_0,
- $\pi_k = \frac{\lambda_{k-1}\lambda_{k-2}\ldots\lambda_0}{\mu_k \mu_{k-1}\ldots\mu_1} \pi_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \pi_0$
- The probability π_0 is determined by the normalization condition π_0
- $\pi_0 = \frac{1}{1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\cdots} = \frac{1}{1+\sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}$

- The time-dependent solution of a BD process
- Above we considered the equilibrium distribution π of a BD process. Sometimes the state probabilities at time 0, $\boldsymbol{\pi}(0)= \{\pi_0(0),\pi_1(0),\ldots\}$, are known
- \- usually one knows that the system at time 0 is precisely in a given state k; then $\pi_k(0) = 1$ and $\pi_j(0) = 0$ when $j \neq k$
- and one wishes to determine how the state probabilities evolve as a function of time $\boldsymbol{\pi}(t)=(\{\pi_0(t),\pi_1(t),\ldots\})$ - in the limit we have $\lim_{t\to\infty} \boldsymbol{\pi}(t) = \boldsymbol{\pi} = \{\pi_0,\pi_1 \ldots\}$
- This is determined by the equation (Kolmogorov's forward ODE i.e. *P'(t) = P(t)Q , here* $\boldsymbol{\pi}'(t) = \boldsymbol{\pi}(t)Q$)
- $\frac{d}{dt}\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t) \cdot \mathbf{Q}$ where

$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 \\ \vdots & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 \\ \vdots & \vdots & 0 & \mu_4 & -(\lambda_4+\mu_4) \end{pmatrix}$$

- The time-dependent solution of a BD process (continued)

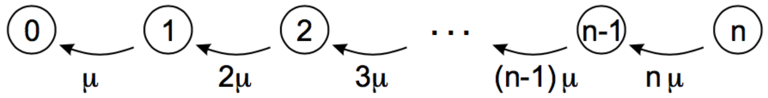


- The equations component wise

- $$\begin{cases} \dfrac{d\pi_i(t)}{dt} = \underbrace{-(\lambda_i + \mu_i)\pi_i(t)}_{\text{flows out}} \underbrace{+\lambda_{i-1}\pi_{i-1}(t) + \mu_{i+1}\pi_{i+1}(t)}_{\text{flows in}} \quad i = 1, 2, \ldots \\ \dfrac{d\pi_0(t)}{dt} = \underbrace{-\lambda_0\pi_0(t)}_{\text{flow out}} \underbrace{+\mu_1\pi_1(t)}_{\text{flow in}} \end{cases}$$

- Example 1. Pure death process

$\begin{cases} \lambda_i = 0 \\ \mu_i = i\mu \end{cases}$ i= 0,1,2,… $\pi_i = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$ all individuals have the same mortality rate μ the system starts from state n



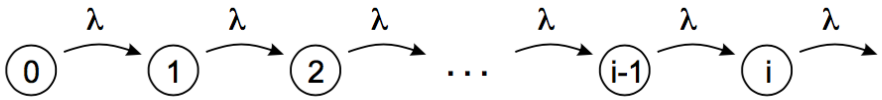
State 0 is an absorbing state, other states are transient

- $\begin{cases} \frac{d}{dt}\pi_n(t) = -n\mu\pi_n(t) & \Longrightarrow \pi_n(t) = e^{-n\mu t} \\ \frac{d}{dt}\pi_i(t) = (i+1)\mu\pi_{i+1}(t) - i\mu\pi_i(t) & i = 0,1,\ldots,n-1 \end{cases}$ multiply the 2nd by $e^{i\mu t}$
- $\frac{d}{dt}\left(e^{i\mu t}\pi_i(t)\right) = (i+1)\mu\pi_{i+1}(t)e^{i\mu t} \Rightarrow \pi_i(t) = (i+1)e^{-i\mu t}\mu\int_0^t \pi_{i+1}(t')\, e^{i\mu t'}dt'$
- $\pi_{n-1}(t) = ne^{-(n-1)\mu t}\mu\int_0^t \pi_n(t')e^{(n-1)\mu t'}dt' = ne^{-(n-1)\mu t}\mu\int_0^t e^{-n\mu t'}e^{(n-1)\mu t'}dt' = ne^{-(n-1)\mu t}(1-e^{-\mu t})$
- Recursively $\pi_i(t) = \binom{n}{i}(e^{-\mu t})^i(1-e^{-\mu t})^{n-i}$ ~ Bin(n, $e^{-\mu t}$) : $e^{-\mu t}$=p in Bin(n,p) is the survival probability at time t independent of others

- Example 2. Pure birth process (Poisson process) below we view the Poisson process as counting process

- $\begin{cases} \lambda_i = \lambda \\ \mu_i = 0 \end{cases}$ i= 0,1,2,… $\pi_i(0) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}$

  birth probability per time unit is constant λ initially the population size is 0



All states are transient

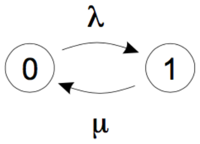
- $\begin{cases} \frac{d}{dt}\pi_i(t) = -\lambda\pi_i(t) + \lambda\pi_{i-1}(t) & i > 0 \\ \frac{d}{dt}\pi_0(t) = -\lambda\pi_0(t) & \Rightarrow \quad \pi_0(t) = e^{-\lambda t} \end{cases}$ multiply the 1st by $e^{\lambda t}$

- $\frac{d}{dt}(e^{\lambda t}\pi_i(t)) = \lambda\pi_{i-1}(t)e^{\lambda t} \Rightarrow \quad \pi_i(t) = e^{-\lambda t}\lambda \int_0^t \pi_{i-1}(t') \, e^{\lambda t'} dt'$

- $\pi_1(t) = e^{-\lambda t}\lambda \int_0^t e^{-\lambda t'} \, e^{\lambda t'} dt' = e^{-\lambda t}(\lambda t)$

- Recursively $\pi_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$ Number of births in interval (0, t) ~ Poisson(λt)

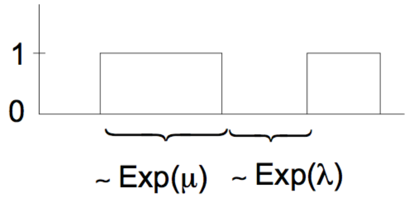
- Example 3. A single server system



- constant arrival rate λ (Poisson arrivals)
- stopping rate of the service μ (exponential distribution)

The states of the system

$$\begin{cases} 0 & \text{server free} \\ 1 & \text{server busy} \end{cases}$$



- $$\begin{cases} \frac{d}{dt}\pi_0(t) = -\lambda\pi_0(t) + \mu\pi_1(t) \\ \frac{d}{dt}\pi_1(t) = \lambda\pi_0(t) - \mu\pi_1(t) \end{cases}$$

Vector differential equation with 2 equations that are depicted on the left

$$\frac{d}{dt}\boldsymbol{\pi}(t) = \boldsymbol{\pi}(t)\mathbf{Q} \Rightarrow \frac{d}{dt}\{\pi_0(t), \pi_1(t)\} = \{\pi_0(t), \pi_1(t)\}Q$$

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

- BY adding both sides of the equations
- $\frac{d}{dt}(\pi_0(t) + \pi_1(t)) = 0 \Rightarrow \pi_0(t) + \pi_1(t) = \text{constant} = 1 \Rightarrow \pi_1(t) = 1 - \pi_0(t)$
- $\frac{d}{dt}\pi_0(t) + (\lambda + \mu)\pi_0(t) = \mu$ (multiply by $e^{(\lambda+\mu)t}$) $\Rightarrow \frac{d}{dt}(e^{(\lambda+\mu)t}\pi_0(t)) = \mu e^{(\lambda+\mu)t} \Rightarrow d(e^{(\lambda+\mu)t}\pi_0(t)) = \mu e^{(\lambda+\mu)t}dt \Rightarrow d(e^{(\lambda+\mu)t}\pi_0(t)) = \mu e^{(\lambda+\mu)t}dt \Rightarrow \int_0^t d(e^{(\lambda+\mu)t'}\pi_0(t')) = \int_0^t \mu e^{(\lambda+\mu)t'}dt' \Rightarrow e^{(\lambda+\mu)t}\pi_0(t) - \pi_0(0) = \frac{\mu}{\lambda+\mu}e^{(\lambda+\mu)t} - \frac{\mu}{\lambda+\mu} \Rightarrow e^{(\lambda+\mu)t}\pi_0(t) = \frac{\mu}{\lambda+\mu}e^{(\lambda+\mu)t} + (\pi_0(0) - \frac{\mu}{\lambda+\mu})$
- $\pi_0(t) = \frac{\mu}{\lambda+\mu} + (\pi_0(0) - \frac{\mu}{\lambda+\mu})e^{-(\lambda+\mu)t}$
- $\pi_1(t) = \underbrace{\frac{\lambda}{\lambda+\mu}}_{\text{equilibrium distribution}} + \underbrace{(\pi_1(0) - \frac{\lambda}{\lambda+\mu})}_{\text{deviation from the equilibrium}}\underbrace{e^{-(\lambda+\mu)t}}_{\text{decays exponentially}}$

Summary of the analysis on Markov processes

1. Find the state description of the system

• no ready recipe

• often an appropriate description is obvious

• sometimes requires more thinking

• a system may Markovian with respect to one state description but non-Markovian with respect to another one (the state information is not sufficient)

• finding the state description is the creative part of the problem

2. Determine the state transition rates

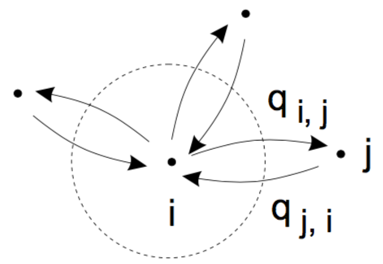
• a straight forward task when holding times and interarrival times are exponential

3. Solve the balance equations

• in principle straight forward (solution of a set of linear equations)
• the number of unknowns (number of states) can be very great
• often the special structure of the transition diagram can be exploited

Appendix

Global balance



$n + 1$ states

$$\boxed{\sum_{j \neq i} \pi_j q_{j,i} = \sum_{j \neq i} \pi_i q_{i,j}}$$

$\underbrace{\sum_{j \neq i} \pi_j q_{j,i}}_{\text{flow to state } i}$, $\underbrace{\sum_{j \neq i} \pi_i q_{i,j}}_{\text{flow out of state } i}$

$i = 0, 1, \ldots, n$
one equation per each state

$$\overbrace{(\pi_0, \ldots, \pi_n)}^{\boldsymbol{\pi}} \overbrace{\begin{pmatrix} -\sum_j q_{0,j} & q_{0,1} & q_{0,2} & \cdots & q_{0,n} \\ q_{1,0} & -\sum_j q_{1,j} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,0} & q_{2,1} & -\sum_j q_{2,j} & \cdots & q_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,0} & q_{n,1}, & q_{n,2}, & \cdots & -\sum_j q_{n,j} \end{pmatrix}}^{\mathbf{Q}} = \begin{pmatrix} 0 \\ 0 \\ \\ \vdots \\ 0 \end{pmatrix}$$

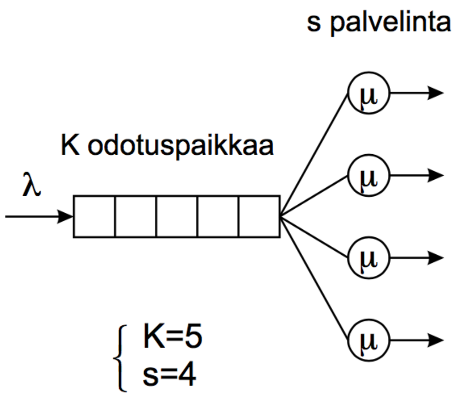
$$\boxed{\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0}}$$

one equation is redundant

$$\boxed{\pi_0 + \pi_1 + \cdots + \pi_n = 1}$$

normalization condition

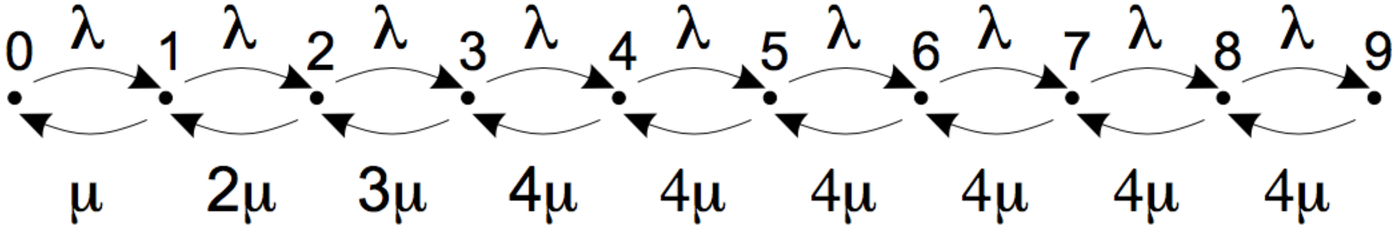
Example 1. A queueing system



- s servers
- K waiting places
- λ arrival rate (Poisson)
- μ Exp(μ) holding time (expectation $1/\mu$)

The number of customers in system N is an appropriate state variable
- uniquely determines the number of customers in service and in waiting room

- after each arrival and departure the remaining service times of the customers in service are Exp(μ) distributed (memoryless)

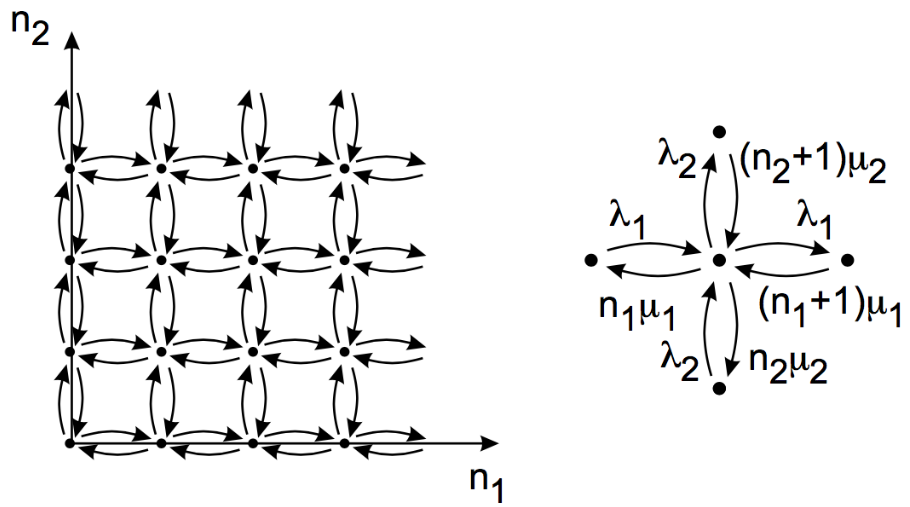


Example 2. Call blocking in an ATM network

A virtual path (VP) of an ATM network is offered calls of two different types.

$$\begin{cases} R_1 = 1\text{Mbps} \\ \lambda_1 = \text{arrival rate} \\ \mu_1 = \text{mean holding time} \end{cases} \qquad \begin{cases} R_2 = 2\text{Mbps} \\ \lambda_2 = \text{arrival rate} \\ \mu_2 = \text{mean holding time} \end{cases}$$

a) The capacity of the link is large (infinite)



The state variable of the Markov process in this example is the pair (N1, N2), where Ni defines the number of class-i connections in progress.

Call blocking in an ATM network (continued)

b) The capacity of the link is 4.5 Mbps